A Technique for Verifying Measurements

Keye Martin[1] and Prakash Panangaden[2]

[1]Center for High Assurance Computer Systems
U. S. Naval Research Laboratory
[2]School of Computer Science
McGill University

25th May 2008 / Mathematical Foundations Of Programming Semantics

Outline

Outline

Outline

Outline

Outline

Summary

- What do measurements tell us?
- Formal definitions
- A trivial lemma
- The main theorem
- An example from relativity

Summary

- What do measurements tell us?
- Formal definitions
- A trivial lemma
- The main theorem
- An example from relativity

Summary

- What do measurements tell us?
- Formal definitions
- A trivial lemma
- The main theorem
- An example from relativity

Summary

- What do measurements tell us?
- Formal definitions
- A trivial lemma
- The main theorem
- An example from relativity

Summary

- What do measurements tell us?
- Formal definitions
- A trivial lemma
- The main theorem
- An example from relativity

Intuitions

- Domain theory formalizes notions of partial information and computability as processing of finite pieces of information.
- Standard domain theory tells us that Scott-continuous functions on dcpos have least fixed points.
- Some non-Scott-continuous functions, however, seem to have fixed points anyway: zero-finding. Why?
- A quantitative *measure* of the partiality *is* continuous.

Intuitions

- Domain theory formalizes notions of partial information and computability as processing of finite pieces of information.
- Standard domain theory tells us that Scott-continuous functions on dcpos have least fixed points.
- Some non-Scott-continuous functions, however, seem to have fixed points anyway: zero-finding. Why?
- A quantitative *measure* of the partiality *is* continuous.

Intuitions

- Domain theory formalizes notions of partial information and computability as processing of finite pieces of information.
- Standard domain theory tells us that Scott-continuous functions on dcpos have least fixed points.
- Some non-Scott-continuous functions, however, seem to have fixed points anyway: zero-finding. Why?
- A quantitative *measure* of the partiality *is* continuous.

Intuitions

- Domain theory formalizes notions of partial information and computability as processing of finite pieces of information.
- Standard domain theory tells us that Scott-continuous functions on dcpos have least fixed points.
- Some non-Scott-continuous functions, however, seem to have fixed points anyway: zero-finding. Why?
- A quantitative *measure* of the partiality *is* continuous.

History

- Invented by Keye Martin in his PhD thesis (2000).
- Similar structure found in domains of classical and quantum states (Coecke and Martin, 2001-2002).
- Domains of communication channels were found to have marvelous algebraic, geometric and informatic structure which made impact on information theory (Martin, Allwein, Moskowitz, Chatzikokolakis, 2005-2008)
- Spacetime has domain theoretic structure which ties causality and topology together (Martin and Panangaden, 2004-2006)
- There are *measurements* that incorporate the geometry of spacetime. (Martin and Panangaden 2007-??).

History

- Invented by Keye Martin in his PhD thesis (2000).
- Similar structure found in domains of classical and quantum states (Coecke and Martin, 2001-2002).
- Domains of communication channels were found to have marvelous algebraic, geometric and informatic structure which made impact on information theory (Martin, Allwein, Moskowitz, Chatzikokolakis, 2005-2008)
- Spacetime has domain theoretic structure which ties causality and topology together (Martin and Panangaden, 2004-2006)
- There are *measurements* that incorporate the geometry of spacetime. (Martin and Panangaden 2007-??).

History

- Invented by Keye Martin in his PhD thesis (2000).
- Similar structure found in domains of classical and quantum states (Coecke and Martin, 2001-2002).
- Domains of communication channels were found to have marvelous algebraic, geometric and informatic structure which made impact on information theory (Martin, Allwein, Moskowitz, Chatzikokolakis, 2005-2008)
- Spacetime has domain theoretic structure which ties causality and topology together (Martin and Panangaden, 2004-2006)
- There are *measurements* that incorporate the geometry of spacetime. (Martin and Panangaden 2007-??).

History

- Invented by Keye Martin in his PhD thesis (2000).
- Similar structure found in domains of classical and quantum states (Coecke and Martin, 2001-2002).
- Domains of communication channels were found to have marvelous algebraic, geometric and informatic structure which made impact on information theory (Martin, Allwein, Moskowitz, Chatzikokolakis, 2005-2008)
- Spacetime has domain theoretic structure which ties causality and topology together (Martin and Panangaden, 2004-2006)
- There are *measurements* that incorporate the geometry of spacetime. (Martin and Panangaden 2007-??).

History

- Invented by Keye Martin in his PhD thesis (2000).
- Similar structure found in domains of classical and quantum states (Coecke and Martin, 2001-2002).
- Domains of communication channels were found to have marvelous algebraic, geometric and informatic structure which made impact on information theory (Martin, Allwein, Moskowitz, Chatzikokolakis, 2005-2008)
- Spacetime has domain theoretic structure which ties causality and topology together (Martin and Panangaden, 2004-2006)
- There are *measurements* that incorporate the geometry of spacetime. (Martin and Panangaden 2007-??).

Root finding 1

Given $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous, define $\mathrm{split}_f : C(f) \longrightarrow C(f)$ by

$$\mathrm{split}_f[a,b] = \begin{cases} \mathrm{left}[a,b] & \text{if } \mathrm{left}[a,b] \in C(f) \\ \mathrm{right}[a,b] & \text{otherwise.} \end{cases}$$

$C(f)$ is the set of intervals where f changes sign and “left” and “right” have the evident meanings. The fixed point of split_f is the root of f.

Root finding 2

- Unfortunately, split_f is not Scott continuous
- but, the length of the intervals decreases continuously.
- The length of an interval measures how “partial” it is.

Root finding 2

- Unfortunately, split_f is not Scott continuous
- but, the length of the intervals decreases continuously.
- The length of an interval measures how “partial” it is.

Root finding 2

- Unfortunately, split_f is not Scott continuous
- but, the length of the intervals decreases continuously.
- The length of an interval measures how “partial” it is.

Measuring the content of an $x \in D$

A Scott continuous map $\mu : D \longrightarrow E$ between posets D, E is said to *measure the content* of $x \in D$ if

$$x \in U \Rightarrow (\exists \epsilon \in \sigma_E) x \in \mu_\epsilon \subseteq U,$$

whenever $U \in \sigma_D$ is Scott open and

$$\mu_\epsilon(x) := \mu^{-1}(\epsilon) \cap \downarrow x$$

are the elements ϵ-close to x.

An important special case

Take E to be $[0, \infty)^*$. Then the definition of μ measuring x in D is: for all Scott open sets $U \subseteq D$,

$$x \in U \Rightarrow (\exists \epsilon > 0) x \in \mu_\epsilon(x) \subseteq U$$

where

$$\mu_\epsilon(x) := \{y \in D : y \sqsubseteq x \text{ and } |\mu x - \mu y| < \epsilon\}$$

are the ϵ-approximations of x.

Measurement

A *measurement* $\mu : D \longrightarrow [0, \infty)^*$ is a Scott continuous map that measures all the xs in $\ker(\mu) := \{x \in D : \mu x = 0\}$.
The fact that it is not *required* to measure all of D means that measurements are more widely applicable than they would be otherwise.

Who cares?

Suppose that we have an approximating sequence (x_n) for $x \in D$ and we want to know when we are “close enough” according to some $a \ll x$. The measurement tells us when we are close enough.

$$(\exists \epsilon > 0)(\forall n)(x_n \sqsubseteq x \text{ and } |\mu x - \mu x_n| < \epsilon) \Rightarrow a \ll x_n.$$

Since $\uparrow\!\!\!\uparrow a$ forms a basis for the Scott topology one can work with any Scott open.

Measuring the whole domain

- When does a measurement measure the whole domain?
- There are several examples (information theory, entropy of classical states) where one can show that one has a measurement that measures the maximal elements
- and that it is strictly monotone.
- Showing that it measures the whole domain seemed hard to prove.
- Physical intuition suggests that in several examples we do have a measurement that measures the whole domain.

Measuring the whole domain

- When does a measurement measure the whole domain?
- There are several examples (information theory, entropy of classical states) where one can show that one has a measurement that measures the maximal elements
- and that it is strictly monotone.
- Showing that it measures the whole domain seemed hard to prove.
- Physical intuition suggests that in several examples we do have a measurement that measures the whole domain.

Measuring the whole domain

- When does a measurement measure the whole domain?
- There are several examples (information theory, entropy of classical states) where one can show that one has a measurement that measures the maximal elements
- and that it is strictly monotone.
- Showing that it measures the whole domain seemed hard to prove.
- Physical intuition suggests that in several examples we do have a measurement that measures the whole domain.

Measuring the whole domain

- When does a measurement measure the whole domain?
- There are several examples (information theory, entropy of classical states) where one can show that one has a measurement that measures the maximal elements
- and that it is strictly monotone.
- Showing that it measures the whole domain seemed hard to prove.
- Physical intuition suggests that in several examples we do have a measurement that measures the whole domain.

Measuring the whole domain

- When does a measurement measure the whole domain?
- There are several examples (information theory, entropy of classical states) where one can show that one has a measurement that measures the maximal elements
- and that it is strictly monotone.
- Showing that it measures the whole domain seemed hard to prove.
- Physical intuition suggests that in several examples we do have a measurement that measures the whole domain.

Trivial Lemma

For a sequence (x_n) in a compact Hausdorff space X, the following are equivalent:
(i) The sequence (x_i) converges to x.
(ii) For any convergent subsequence (x_{n_k}) of (x_n), we have that $(x_{n_k}) \longrightarrow x$.
Amazingly, this is a key step of the next proposition.

Theorem: Let $\mu : D \longrightarrow [0, \infty)^*$ be a strictly monotone, Scott-continuous function defined on a poset D. If τ is a Hausdorff topology such that

1. τ contains the Scott topology,
2. every sequence (x_n) in $\downarrow x$ with $\mu x_n \longrightarrow \mu x$ is contained in some compact $K \subseteq \downarrow x$,
3. the function μ is continuous from (D, τ) to $[0, \infty)$ with the Euclidean topology,

then μ measures all of D.
Corollary: one can drop (ii) if τ is compact since then $\downarrow x$ is τ-compact.

Theorem: Let $\mu : D \longrightarrow [0, \infty)^*$ be a strictly monotone, Scott-continuous function defined on a poset D. If τ is a Hausdorff topology such that

1. τ contains the Scott topology,
2. every sequence (x_n) in $\downarrow x$ with $\mu x_n \longrightarrow \mu x$ is contained in some compact $K \subseteq \downarrow x$,
3. the function μ is continuous from (D, τ) to $[0, \infty)$ with the Euclidean topology,

then μ measures all of D.
Corollary: one can drop (ii) if τ is compact since then $\downarrow x$ is τ-compact.

Theorem: Let $\mu : D \longrightarrow [0, \infty)^*$ be a strictly monotone, Scott-continuous function defined on a poset D. If τ is a Hausdorff topology such that

1. τ contains the Scott topology,
2. every sequence (x_n) in $\downarrow x$ with $\mu x_n \longrightarrow \mu x$ is contained in some compact $K \subseteq \downarrow x$,
3. the function μ is continuous from (D, τ) to $[0, \infty)$ with the Euclidean topology,

then μ measures all of D.
Corollary: one can drop (ii) if τ is compact since then $\downarrow x$ is τ-compact.

Global hyperbolicity

- We showed that a certain class of posets – globally hyperbolic – can be given the structure of *interval domains*.
- These posets arise naturally in the study of causal structure of spacetimes. In a GH spacetime the intervals $J^+(a) \cap J^-(b)$ are compact. They are "approximations" to points in spacetime.
- Using a variation of ideal completion we can reconstruct spacetime and its topology from a countable dense subset and the causal order.
- Can we reconstruct the geometry?

Global hyperbolicity

- We showed that a certain class of posets – globally hyperbolic – can be given the structure of *interval domains*.
- These posets arise naturally in the study of causal structure of spacetimes. In a GH spacetime the intervals $J^+(a) \cap J^-(b)$ are compact. They are “approximations” to points in spacetime.
- Using a variation of ideal completion we can reconstruct spacetime and its topology from a countable dense subset and the causal order.
- Can we reconstruct the geometry?

Global hyperbolicity

- We showed that a certain class of posets – globally hyperbolic – can be given the structure of *interval domains*.
- These posets arise naturally in the study of causal structure of spacetimes. In a GH spacetime the intervals $J^+(a) \cap J^-(b)$ are compact. They are "approximations" to points in spacetime.
- Using a variation of ideal completion we can reconstruct spacetime and its topology from a countable dense subset and the causal order.
- Can we reconstruct the geometry?

Global hyperbolicity

- We showed that a certain class of posets – globally hyperbolic – can be given the structure of *interval domains*.
- These posets arise naturally in the study of causal structure of spacetimes. In a GH spacetime the intervals $J^+(a) \cap J^-(b)$ are compact. They are “approximations” to points in spacetime.
- Using a variation of ideal completion we can reconstruct spacetime and its topology from a countable dense subset and the causal order.
- Can we reconstruct the geometry?

Measuring spacetime intervals

- We want a "size" for spacetime intervals.
- We can try to use the volume or the length of the shortest timelike geodesic as a measurement, but this does not work!
- There are non-maximal intervals that have zero value for any Lorentz invariant quantity: the null intervals.
- We are forced to use "global time," a completely non Lorentz-invariant quantity.
- To show that one really gets a measurement we need the theorem.
- In fact the theorem was discovered while trying to prove this fact.

Measuring spacetime intervals

- We want a “size” for spacetime intervals.
- We can try to use the volume or the length of the shortest timelike geodesic as a measurement, but this does not work!
- There are non-maximal intervals that have zero value for any Lorentz invariant quantity: the null intervals.
- We are forced to use “global time,” a completely non Lorentz-invariant quantity.
- To show that one really gets a measurement we need the theorem.
- In fact the theorem was discovered while trying to prove this fact.

Measuring spacetime intervals

- We want a “size” for spacetime intervals.
- We can try to use the volume or the length of the shortest timelike geodesic as a measurement, but this does not work!
- There are non-maximal intervals that have zero value for any Lorentz invariant quantity: the null intervals.
- We are forced to use “global time,” a completely non Lorentz-invariant quantity.
- To show that one really gets a measurement we need the theorem.
- In fact the theorem was discovered while trying to prove this fact

Measuring spacetime intervals

- We want a “size” for spacetime intervals.
- We can try to use the volume or the length of the shortest timelike geodesic as a measurement, but this does not work!
- There are non-maximal intervals that have zero value for any Lorentz invariant quantity: the null intervals.
- We are forced to use “global time,” a completely non Lorentz-invariant quantity.
- To show that one really gets a measurement we need the theorem.
- In fact the theorem was discovered while trying to prove this fact

Measuring spacetime intervals

- We want a “size” for spacetime intervals.
- We can try to use the volume or the length of the shortest timelike geodesic as a measurement, but this does not work!
- There are non-maximal intervals that have zero value for any Lorentz invariant quantity: the null intervals.
- We are forced to use “global time,” a completely non Lorentz-invariant quantity.
- To show that one really gets a measurement we need the theorem.
- In fact the theorem was discovered while trying to prove this fact

Measuring spacetime intervals

- We want a “size” for spacetime intervals.
- We can try to use the volume or the length of the shortest timelike geodesic as a measurement, but this does not work!
- There are non-maximal intervals that have zero value for any Lorentz invariant quantity: the null intervals.
- We are forced to use “global time,” a completely non Lorentz-invariant quantity.
- To show that one really gets a measurement we need the theorem.
- In fact the theorem was discovered while trying to prove this fact.

Can it be used in other places?

Yes! In the paper we show that capacity measures the whole domain of binary channels and entropy measures the whole domain of classical states. In both cases the main theorem is the main tool.

Can we always use it?

- Yes!! (In principle).
- **Theorem** If μ measures a poset D, then μ is strictly monotone and there is a Hausdorff topology τ satisfying the conditions (i)-(iii) of the previous theorem.
- Take, for τ, the topology with the following basis:

$$\mathcal{B} := \{U \cap V : UScottopen, VScottclosed\}$$

- This topology – the μ topology – is zero dimensional, Hausdorff and contains the Scott topology.

Can we always use it?

- Yes!! (In principle).
- **Theorem** If μ measures a poset D, then μ is strictly monotone and there is a Hausdorff topology τ satisfying the conditions (i)-(iii) of the previous theorem.
- Take, for τ, the topology with the following basis:

$$\mathcal{B} := \{U \cap V : U Scott open, V Scott closed\}$$

- This topology – the μ topology – is zero dimensional, Hausdorff and contains the Scott topology.

Can we always use it?

- Yes!! (In principle).
- **Theorem** If μ measures a poset D, then μ is strictly monotone and there is a Hausdorff topology τ satisfying the conditions (i)-(iii) of the previous theorem.
- Take, for τ, the topology with the following basis:

$$\mathcal{B} := \{U \cap V : UScottopen, VScottclosed\}$$

- This topology – the μ topology – is zero dimensional, Hausdorff and contains the Scott topology.

Can we always use it?

- Yes!! (In principle).
- **Theorem** If μ measures a poset D, then μ is strictly monotone and there is a Hausdorff topology τ satisfying the conditions (i)-(iii) of the previous theorem.
- Take, for τ, the topology with the following basis:

$$\mathcal{B} := \{U \cap V : UScottopen, VScottclosed\}$$

- This topology – the μ topology – is zero dimensional, Hausdorff and contains the Scott topology.