Labelled Markov Processes

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- Wait!! What are Labelled Transition Systems?

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Labelled Transition Systems

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- The transitions can be nondeterministic.
- Intended to model communication and concurrency; the notion of observation is very different from what one uses in automata theory.
- We do not see the states, we see the actions and we observe when actions are rejected by the system.

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- All probabilistic data is *internal* - no probabilities associated with environment behaviour.
- We observe the interactions - not the internal states.
- **In general, the state space of a labelled Markov process may be a *continuum*.**

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Motivation

Model and reason about systems with *continuous* state spaces or continuous time evolution or both.

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- performance modelling,
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- continuous time systems

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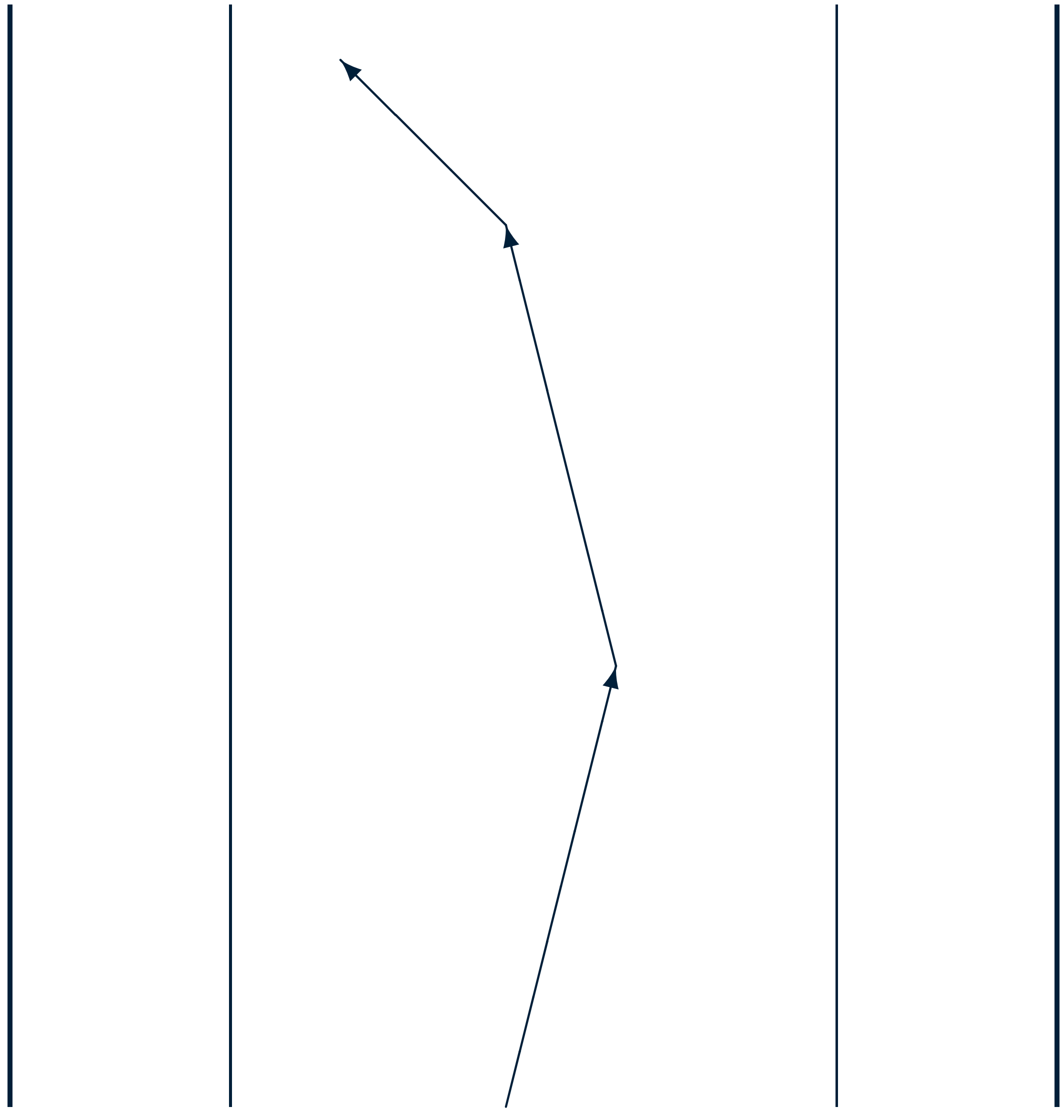
An Example



a - turn left

b - turn right

c - straight

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Actions

a - turn left, b - turn right, c - keep on course

The actions move the craft sideways with some probability distributions on how far it moves. The craft may “drift” even with c. The action a (b) must be disabled when the craft is too near the left (right) boundary.

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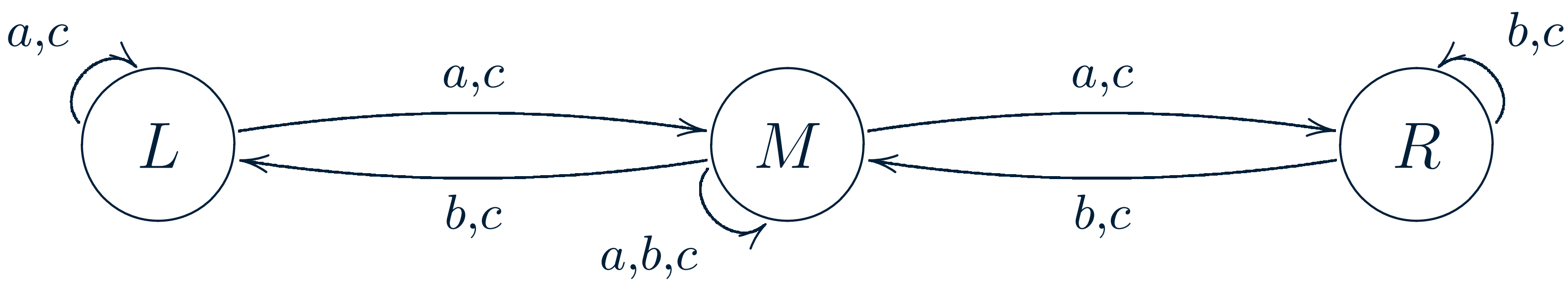
Schematic of Example

- 

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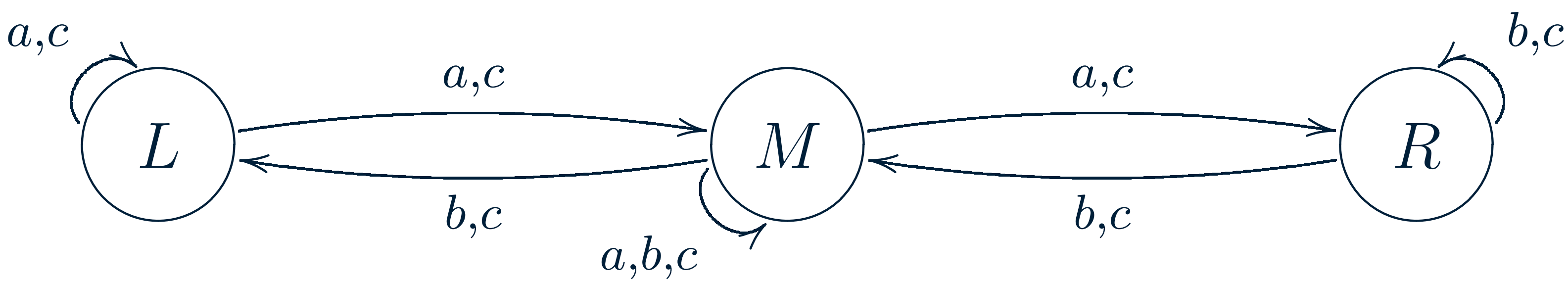
- 

- This picture is misleading: unless very special conditions hold the process cannot be compressed into an *equivalent* (?) finite-state model. In general, the transition probabilities should depend on the position.

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Some remarks on the use of this model

- This is a toy model but exemplifies the issues.
- Can be used for reasoning - much better if we could have a finite-state version.
- Why not discretize right away and never worry about the continuous case? Because we lose the ability to *refine* the model later.
- A better model would be to base it on rewards and think about finiding optimal policies as in AI literature.

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Probabilistic Transition Systems

- A transition system with probabilities and actions (labels) associated with the transitions.
- $$(S, \mathsf{L}, \forall a \in \mathsf{L}\ T_a : S \times S \longrightarrow [0,1])$$
- The model is *reactive*: All probabilistic data is *internal* - no probabilities associated with environment behaviour.

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Examples of PTSs

$$
\begin{tikzcd}
 & s_0 \arrow[dl, "a[\frac{1}{4}]"'] \arrow[dr, "a[\frac{3}{4}]"] & \\
s_1 & & s_2 \arrow[d, "a[1]"] \\
 & & s_3
\end{tikzcd}
$$

A_1

$$
\begin{tikzcd}
 & s_0 \arrow[dl, "a[1]"'] \arrow[dr, "b[1]"] & \\
s_1 \arrow[d, "c[\frac{1}{2}]"'] \arrow[drr, "c[\frac{1}{2}]"] & & s_2 \arrow[d, "a[1]"] \\
s_4 & & s_3
\end{tikzcd}
$$

A_2

When can states be combined?

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- If two states behave in exactly the same way they can be combined.
- In queueing theory there was a notion of lumpability of Markov chains (with no labels).
- In process algebra (with no probabilities) Park and Milner formulated a notion called bisimulation which captures a very fine notion of process equivalence.

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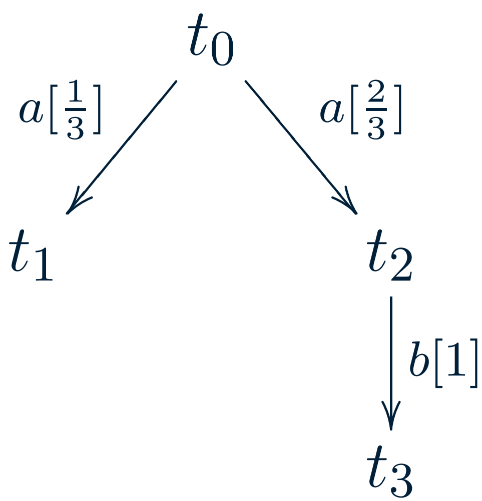
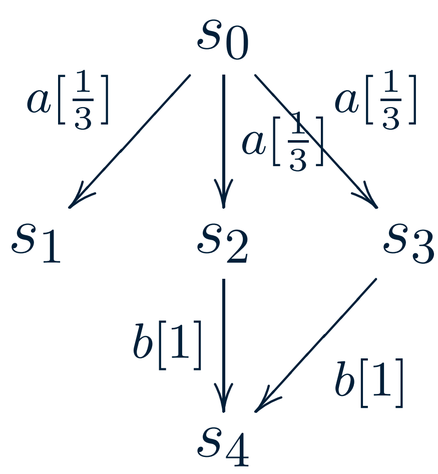
Bisimulation for PTS: Larsen and Skou

- Consider



P_1



P_2

Bisimulation for PTS: Larsen and Skou

- Consider

$$
\begin{tikzcd}
& t_0 \arrow[dl, "{a[\frac{1}{3}]}"'] \arrow[dr, "{a[\frac{2}{3}]}"] & \\
t_1 & & t_2 \arrow[d, "{b[1]}"] \\
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& s_0 \arrow[dl, "{a[\frac{1}{3}]}"'] \arrow[d, "{a[\frac{1}{3}]}"] \arrow[dr, "{a[\frac{1}{3}]}"] & \\
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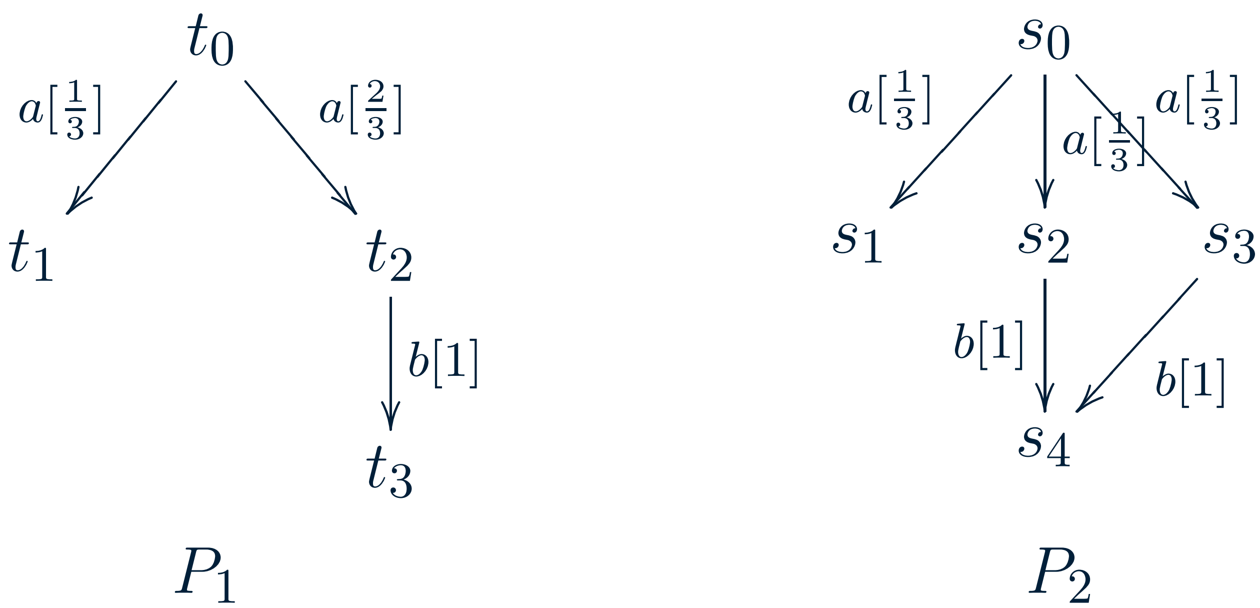

- Should s_0 and t_0 be bisimilar?
- Yes, but we need to add the probabilities.

The Official Definition*

- Let $\mathcal{S} = (S, \mathsf{L}, T_a)$ be a PTS. An equivalence relation R on S is a **bisimulation** if whenever sRs', with $s, s' \in S$, we have that for all $a \in \mathcal{A}$ and every R-equivalence class, A, $T_a(s, A) = T_a(s', A)$.

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- The notation $T_a(s, A)$ means "the probability of starting from s and jumping to a state in the set A."
- Two states are bisimilar if there is some bisimulation relation R relating them.

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The Need for Measure Theory*

- Basic fact: There are subsets of $\mathbf{R}$ for which no sensible notion of size can be defined.

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- Basic fact: There are subsets of $\mathbb{R}$ for which no sensible notion of size can be defined.
- More precisely, there is no translation-invariant measure defined on all the subsets of the reals.

Markov Chains

- A *discrete-time* Markov chain is a finite set S (the state space) together with a transition probability function $T : S \times S \longrightarrow [0, 1]$.

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- A *discrete-time* Markov chain is a finite set S (the state space) together with a transition probability function $T : S \times S \longrightarrow [0, 1]$.
- A Markov chain is just a probabilistic automaton; if we add labels we get a PTS.
- The key property is that the transition probability from s to s' only depends on s and s' and not on the past history of how it got there. This is what allows the probabilistic data to be given as a single matrix T.

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Stochastic Kernels*

- A *stochastic kernel* (Markov kernel) is a function $h : S \times \Sigma \longrightarrow [0, 1]$ with (a) $h(s, \cdot) : \Sigma \longrightarrow [0, 1]$ a (sub)probability measure and (b) $h(\cdot, A) : X \longrightarrow [0, 1]$ a measurable function.

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- Though apparantly asymmetric, these are the stochastic analogues of binary relations
- and the uncountable generalization of a matrix.

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Formal Definition of LMPs*

- An LMP is a tuple $(S, \Sigma, \mathsf{L}, \forall\alpha \in \mathsf{L}.\tau_\alpha)$ where $\tau_\alpha : S \times \Sigma \longrightarrow [0, 1]$ is a *transition probability* function such that

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- $\forall s : S.\lambda A : \Sigma.\tau_\alpha(s, A)$ is a subprobability measure and
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Larsen-Skou Bisimulation*

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  Two states are bisimilar if they are related by a bisimulation relation.
- Can be extended to bisimulation between two different **LMP**s.
- Essentially the same as the version that we had before with zigzag morphisms but much closer in spirit to the Larsen-Skou version.

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Logical Characterization

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$$\exists A \in \Sigma . (\forall s' \in A . s' \models \phi) \wedge (\tau_a(s, A) > q).$$
- Two systems are bisimilar iff they obey the same formulas of $\mathcal{L}$. [DEP 1998 LICS, I and C 2002]

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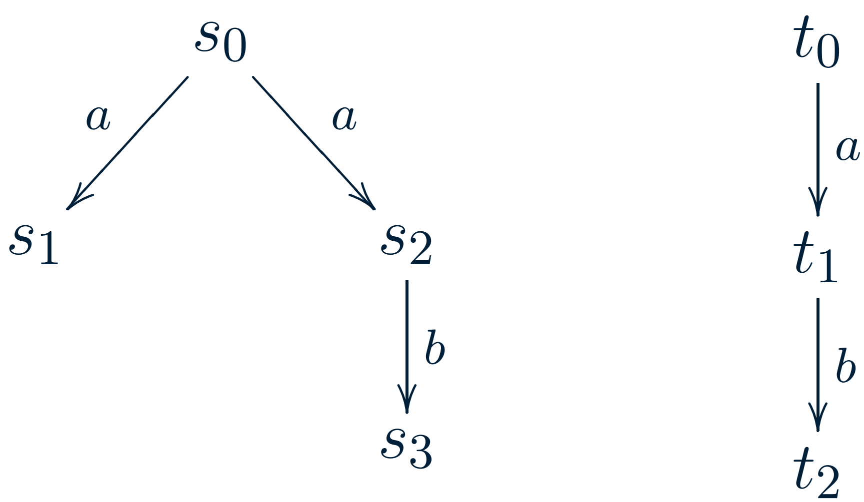
That Cannot be Right?



Two processes that cannot be distinguished without negation.

The formula that distinguishes them is $\langle a\rangle(\neg\langle b\rangle\top)$.

But it is!

- $$\begin{tikzcd}
& s_0 \arrow[dl, "{a[p]}"'] \arrow[dr, "{a[q]}"] & & t_0 \arrow[d, "{a[r]}"] \\
s_1 & & s_2 \arrow[d, "b"] & t_1 \arrow[d, "b"] \\
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\end{tikzcd}$$

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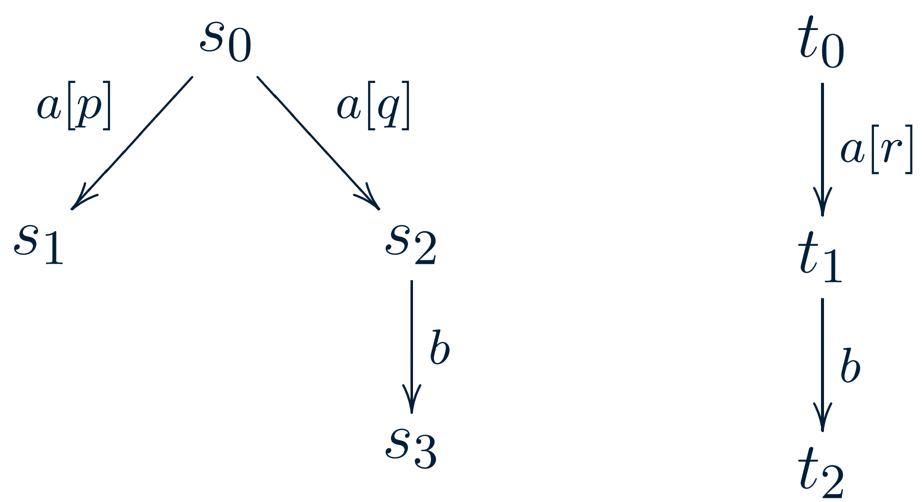
- 

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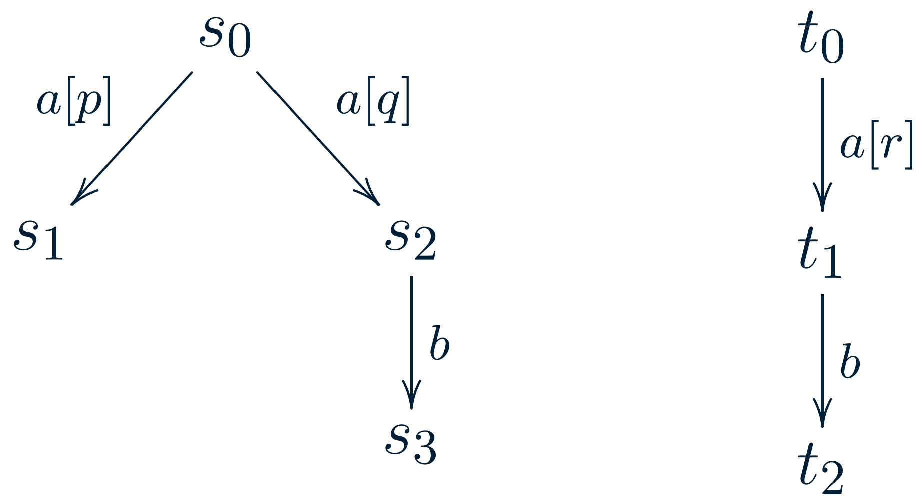
- 

  We add probabilities to the transitions.
- If $p + q < r$ or $p + q > r$ we can easily distinguish them.
- If $p + q = r$ and $p > 0$ then $q < r$ so $\langle a \rangle_r \langle b \rangle_1 \top$ distinguishes them.

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- Use Dynkin's lemma to show that we get a well defined measure on the σ-algebra generated by such sets and the above equality holds.
- Use special properties of analytic spaces to show that this σ-algebra is the same as the original σ-algebra.

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Simulation*

Let $\mathcal{S} = (S, \Sigma, \tau)$ be a labelled Markov process. A preorder R on S is a **simulation** if whenever sRs', we have that for all $a \in \mathcal{A}$ and every R-closed measurable set $A \in \Sigma$, $\tau_a(s, A) \leq \tau_a(s', A)$. We say s is simulated by s' if sRs' for some simulation relation R.

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- The logic used in the characterization has no negation, not even a limited negative construct.
- One can show that if s simulates s' then s satisfies all the formulas of $\mathcal{L}$ that s' satisfies.
- What about the converse?

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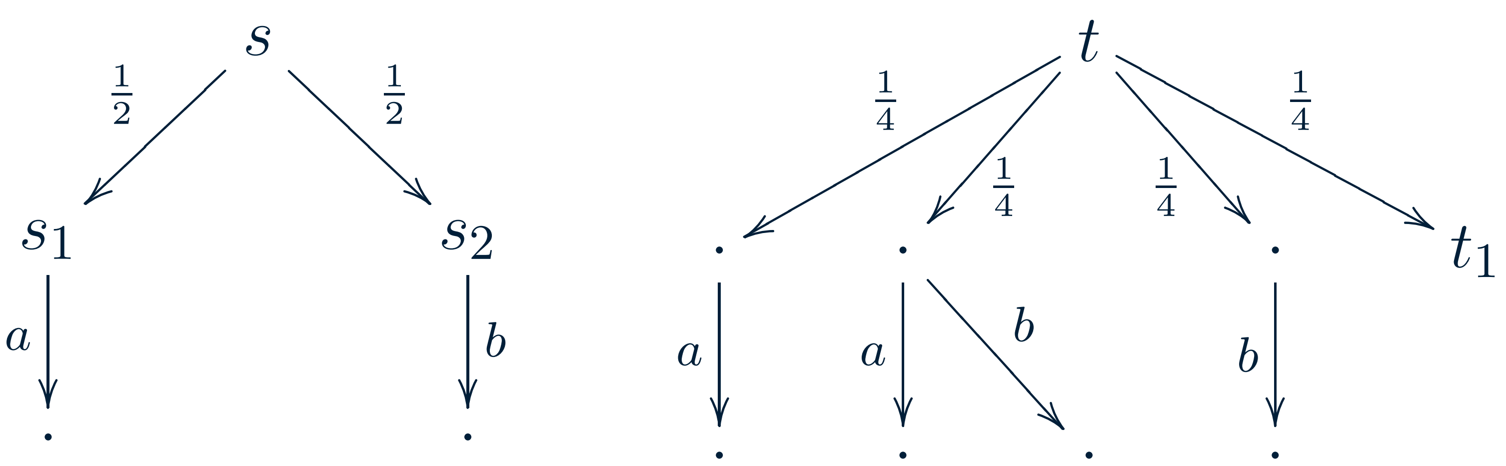
Counter Example!

In the following picture, t satisfies all formulas of $\mathcal{L}$ that s satisfies but t does not simulate s.



All transitions from s and t are labelled by a.

Counter Example (contd.)

- A formula of $\mathcal{L}$ that is satisfied by t but not by s.

$$\langle a \rangle_0 (\langle a \rangle_0 \mathsf{T} \wedge \langle b \rangle_0 \mathsf{T}).$$

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Counter Example (contd.)

- A formula of $\mathcal{L}$ that is satisfied by t but not by s.

$$\langle a \rangle_0 (\langle a \rangle_0 \mathsf{T} \wedge \langle b \rangle_0 \mathsf{T}).$$

- A formula with disjunction that is satisfied by s but not by t:

$$\langle a \rangle_{\frac{3}{4}} (\langle a \rangle_0 \mathsf{T} \vee \langle b \rangle_0 \mathsf{T}).$$

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A Logical Characterization for Simulation

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- With this logic we have:
  An **LMP** s_1 simulates s_2 if and only if for every formula ϕ of $\mathcal{L}_{\vee}$ we have

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- The only proof we know uses domain theory.

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- Exact reasoning is unstable under (small) perturbations of probability numbers.
- Probability numbers are to be viewed as coming with some error estimate: reasoning principles based on the exact value of numbers are of dubious value.
- Probability arises in the modelling of physical systems as an abstraction to specify incomplete knowledge.
- Approximation of probability distributions is often used; e.g. in Monte Carlo schemes.

A metric-based approximate viewpoint

- Move from equality between processes to distances between processes (Jou and Smolka).

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$$d(s,s) = 0, d(s,t) = d(t,s), d(s,u) \leq d(s,t) + d(t,u)$$

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- Move from equality between processes to distances between processes (Jou and Smolka).
- Formalize distance as a metric:

$$d(s,s) = 0, d(s,t) = d(t,s), d(s,u) \leq d(s,t) + d(t,u)$$

- Quantitative measurement of the distinction between processes.

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Criteria on metrics

- Soundness:

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- Stability of distance under temporal evolution:“Nearby states stay close *forever*.”

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Results for Metrics

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- Results work for Markov chains, Labelled Markov processes, Markov decision processes and Labelled Concurrent Markov chains with τ-transitions.
- Establishing closeness of states: Coinduction.
- Distinguishing states: Real-valued modal logics.
- Equational and logical views coincide: Metrics yield same distances as real-valued modal logics.
- Compositional reasoning by *Non-Expansivity*. Process-combinators take closeby processes to closeby processes.

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Approximation Results

- Our main result is a systematic approximation scheme for labeled Markov processes. The set of LMPs is a Polish space. Furthermore, our approximation results allow us to approximate integrals of continuous functions by computing them on finite approximants.

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- Our main result is a systematic approximation scheme for labeled Markov processes. The set of LMPs is a Polish space. Furthermore, our approximation results allow us to approximate integrals of continuous functions by computing them on finite approximants.
- For any LMP, we explicitly provide a (countable) sequence of approximants to it such that:
  - For every logical property satisfied by a process, there is an element of the chain that also satisfies the property.
  - The sequence of approximants converges – in a certain metric – to the process that is being approximated.

Domain-theoretic Results*

- we establish the following equivalence of categories:

$$\mathbf{LMP} \simeq \ \mathit{Proc}$$

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- We show that there is a perfect match between:
  - bisimulation and equality in Proc,
  - simulation and the partial order of Proc,
  - strict simulation and way below in Proc.

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Consequences of the Domain results*

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- The equivalence endows **LMP** with least upper bounds of ω-chains (wrt the simulation ordering). This shows that **LMP** can be used as the target of interpretation of a syntax that includes recursion.

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- We can fix this too with a new approach to approximation based on conditional expectations. [DDP, CONCUR03]

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- In early work on continuous time systems; we gave a logical characterization result for CTMCs [JALP 2003]. Later we showed how to define metrics for a very general class of systems (Generalized Semi-Markov Processes [QEST 2004].
- In more recent work we have shown how to implement the approximation scheme using some Monte Carlo techniques [QEST 2005].