Causality and Order

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Dedicated to Glynn WInskel

Event Structures

- Glynn Winskel's 1981 thesis: Events in Computation was a landmark.
- I saw it in 1982 just as I was moving from Physics (relativity) to Computer Science.
- It, and Lamport's earlier paper, influenced me greatly.

Summary

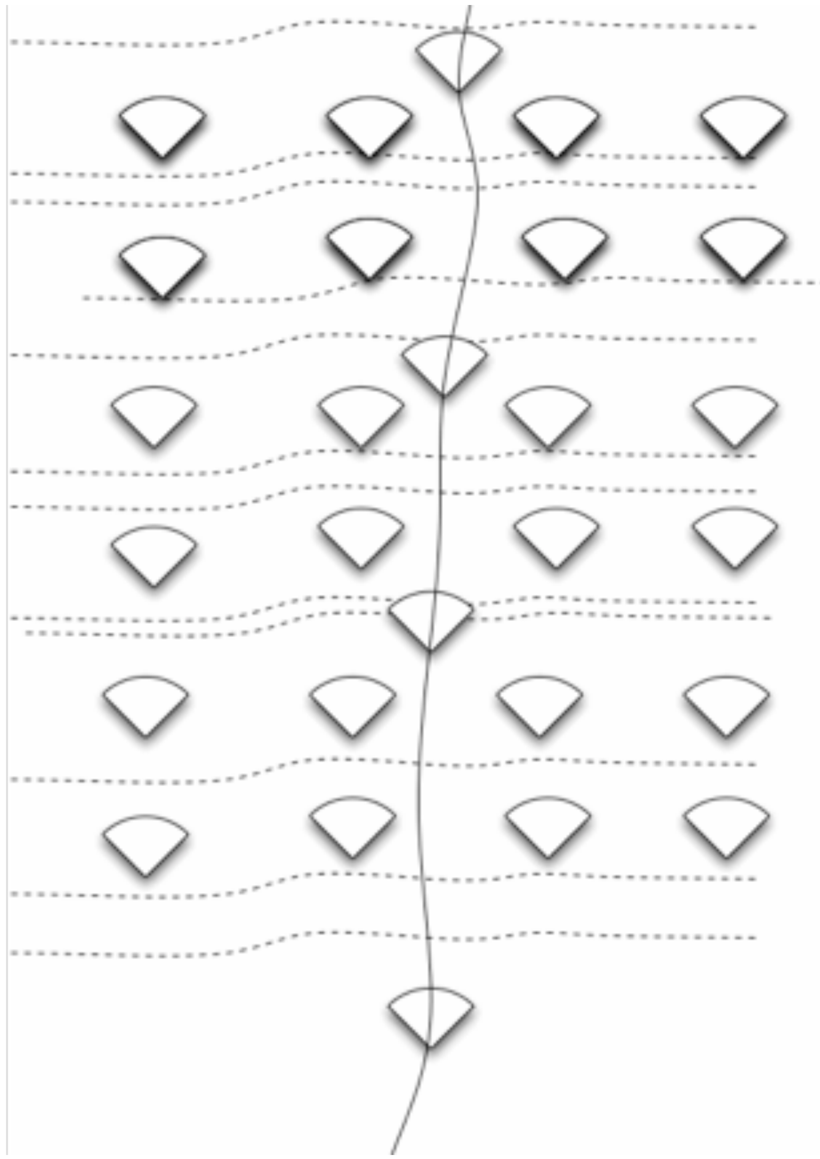
- The Spacetime canvas
- Causality conditions
- Axioms for causal orders
- Differences between causal structure in spacetime and in computation

What is causality?

- The mathematical theory of alibis!
- I did not do it! I could not have done it, I was nowhere nearby!!
- How far do you have to be for this to be a cast iron alibi?
- Spacetime structure has this built in.

Spacetime

- Events
- Light cones
- World lines
- Spatial slices



The Spacetime Canvas

Events: primitive "point like" — they form a set.

Topological space: 4-dimensional manifold;

Hausdorff, locally compact, separable; these imply paracompact.

Smooth structure: now we can define *tangent vectors*.

Spacetime is now equipped with a "smoothly varying" collection of 4-dimensional tangent vector spaces attached to each point: the tangent bundle.

Causal structure: light cones at each point.

A *pair of cones* at each tangent space. The paths of light rays.

Affine connection: how to parallel transport vectors.

Now we can define *geodesics*. The paths of dust particles.

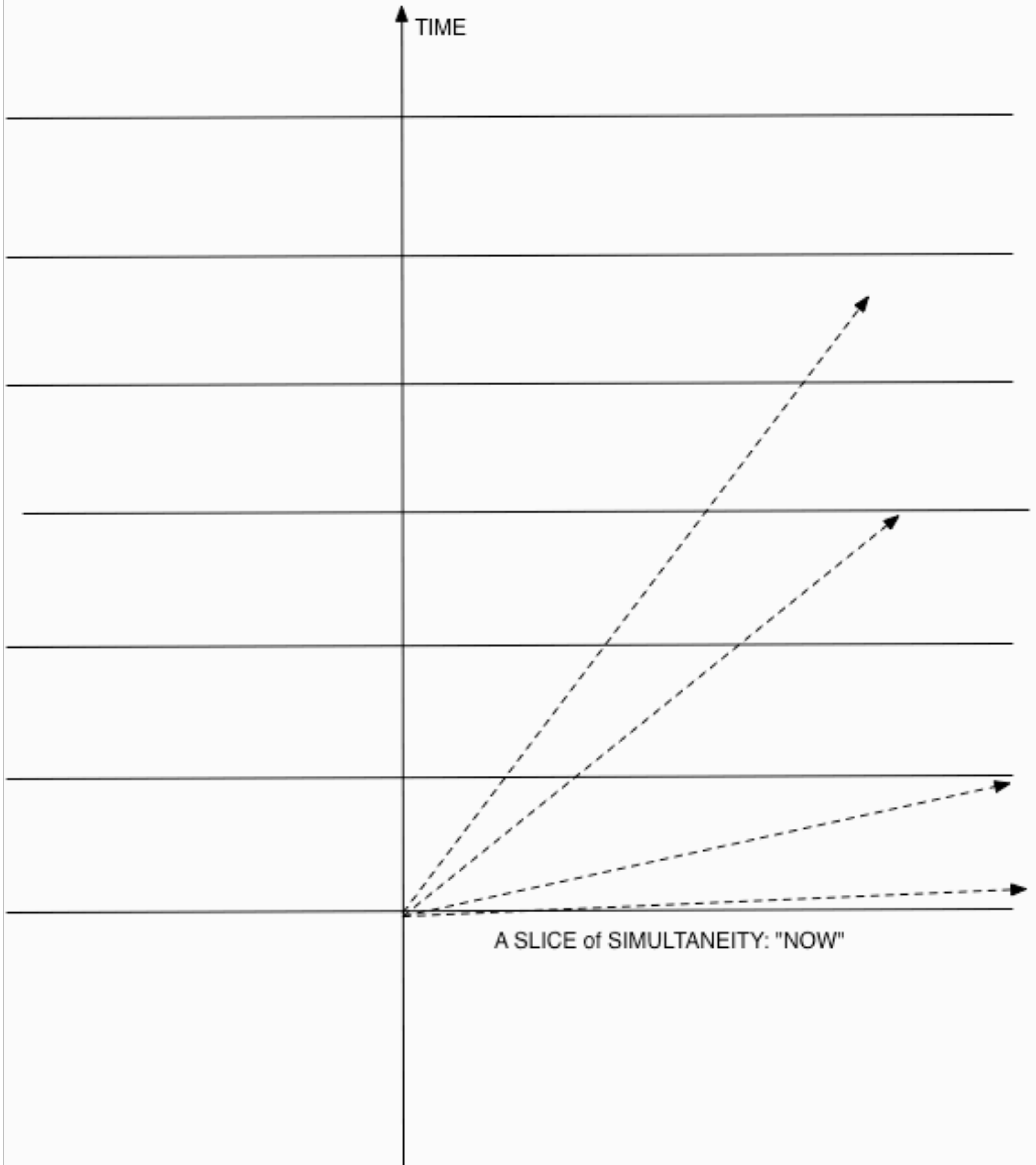
Finishing touches

Now we know the trajectories of dust and light rays.

The only remaining ingredient: the metric.

Usually the metric is given up front and all the other structures are derived from it.

But conceptually, physically and mathematically it comes last.


TIME
A SLICE of SIMULTANEITY: "NOW"

Einstein-Minkowski Causal Structure

Light cones delimit regions of causal influence.

Basic assumption: time-orientability, *i.e.* there is a consistent distinction between future- and past-pointing light cones.

A vector *inside* the light-cone is **timelike.**

A curve whose tangent vector is everywhere future pointing and timelike is called a timelike curve.

A vector on the light cone is **null**.

A vector outside the light cone is **spacelike**.

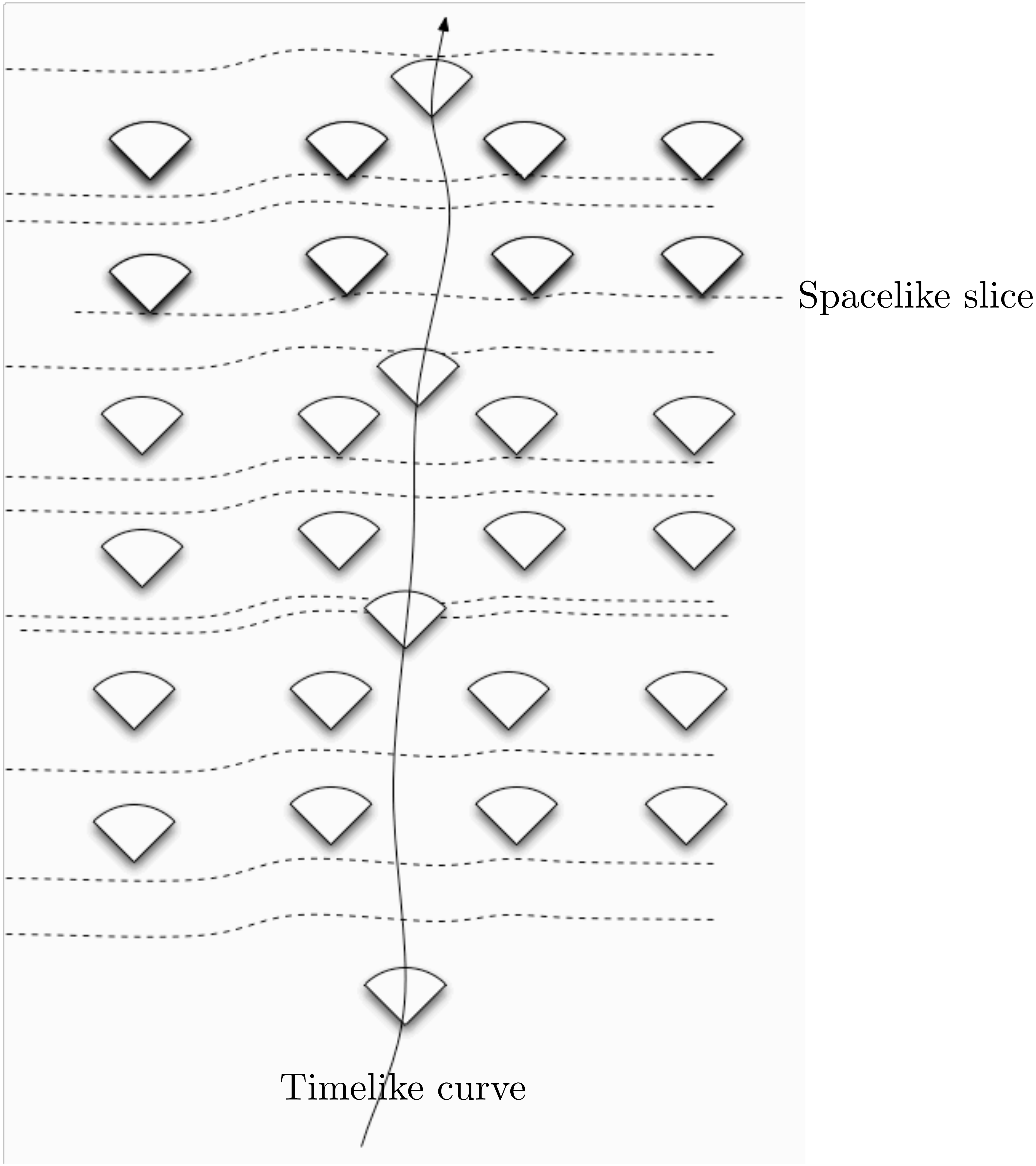
Material particles travel on timelike curves and light travels on null curves.

A spacelike **slice** is a maximal 3-dimensional hypersurface where every two points are spacelike related.

A decomposition of spacetime into a family of disjoint spacelike slices is called a **foliation**.

In Newtonian spacetime there is a *unique, canonical* foliation.

In relativistic spacetimes, there are many possible foliations.


Spacelike slice
Timelike curve

Causal orders

Chronological order, $x \ll y$: there is a smooth timelike curve (with future-pointing tangent) from x to y.
(Regarded as irreflexive.)

Causal order, $x \leq y$: there is a piecewise-smooth curve from x to y with a future-pointing tangent vector that is timelike or null.

Fundamental assumption: $\leq$ is a partial order.

$$J^+(x) := \{y \mid x \leq y\}, J^-(x) := \{y \mid y \leq x\}$$

$$I^+(x) := \{y \mid x \ll y\}, I^-(x) := \{y \mid y \ll x\}$$ "open sets"

Causality Assumptions

1. $x \ll y \Rightarrow y \not\ll x$ Chronology condition

2. $(x \leq y)\ \&\ (y \leq x) \Rightarrow x = y$ Causality condition

Usually, we think that this is enough to rule out causal anomalies.

But, in the continuum more complicated causal anomalies can occur.

There is a hierarchy of increasingly stringent causality conditions that are imposed.

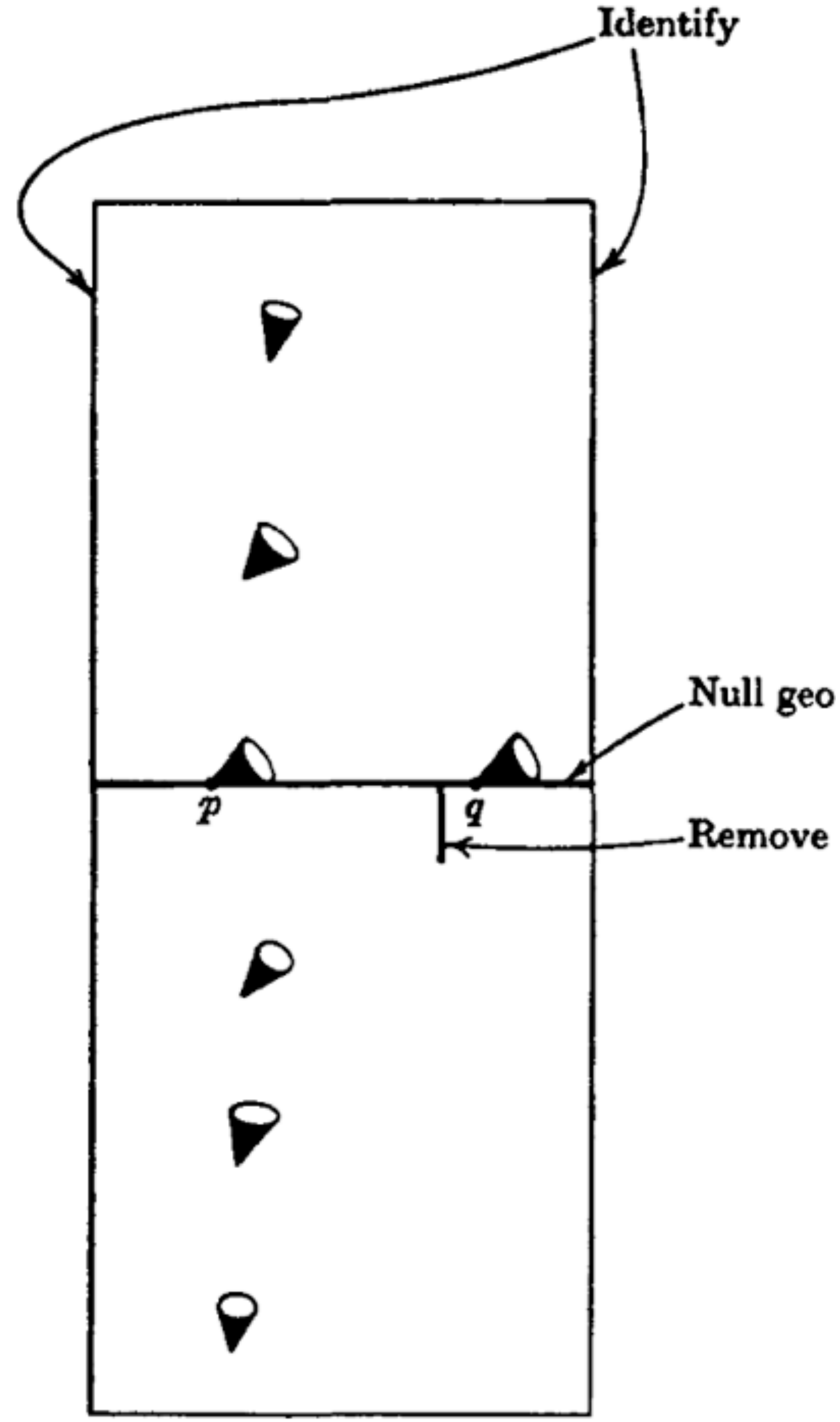
$$\text{Causality} \rightarrow \ldots \rightarrow \text{Global hyperbolicity.}$$

Stricter Assumptions

Future/Past distinguishing: $I^{\pm}(x) = I^{\pm}(y) \Rightarrow x = y.$

There are 3 possible conditions here; they are all different.

This satisfies causality
and past distinguishability
but not future
distinguishability.



The light cones tip over,
become horizontal, and
tip back up. A small
vertical strip has been
removed to break the
closed null curve
that would otherwise occur.
Here $I^+(p) = I^+(q)$.

Stricter Assumptions

Future/Past distinguishing: $I^{\pm}(x) = I^{\pm}(y) \Rightarrow x = y.$

There are 3 possible conditions here; they are all different.

Strong causality at p: Every neighbourhood of p contains a sub-neighbourhood such that no causal curve intersects it more than once.

Imposing strong causality on a compact set means that no causal curve can get imprisoned in this set.

It is possible to have a spacetime satisfying the previous conditions but violating strong causality.

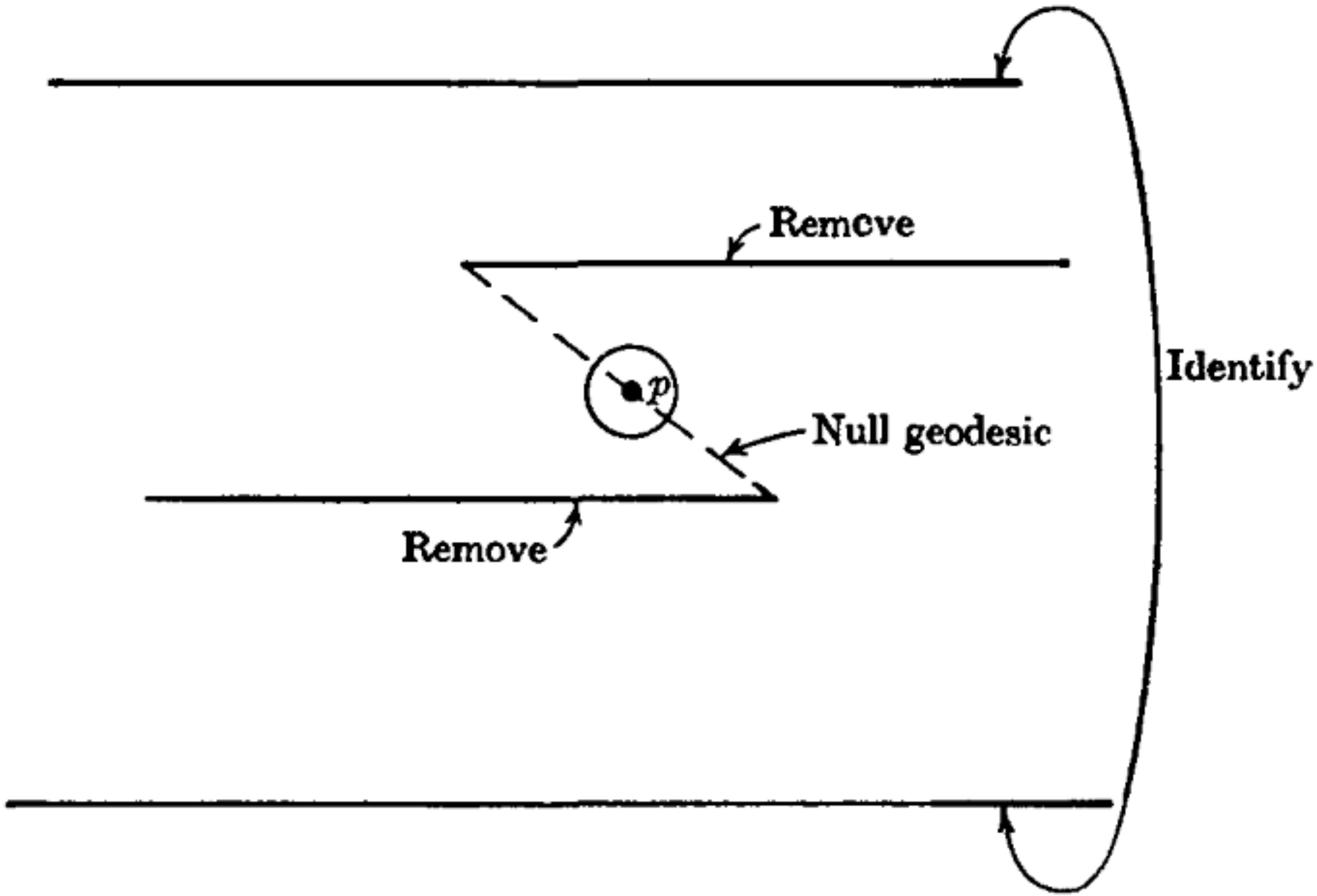


FIGURE 38. A space–time satisfying the causality, future and past distinguishing conditions, but not satisfying the strong causality condition at p. Two strips have been removed from a cylinder; light cones are at $\pm 45°$.

Picture taken from: *The Large-Scale Structure of Spacetime* by Hawking and Ellis.

Consequences of Strong Causality

Define $\ll x, y \gg = I^+(x) \cap I^-(y)$.

If $\ll x, y \gg \cap \ll p, q \gg \neq \emptyset$ then there are u, v such that $\ll u, v \gg \subset \ll x, y \gg \cap \ll p, q \gg$.

Use these sets as the base for a topology:
the Alexandrov topology.

Penrose discovered a strong link between strong causality and the manifold topology.

Penrose's Theorem

Theorem(Penrose) The following are equivalent:

1. M is strongly causal,
2. The Alexandrov topology agrees with the manifold topology,
3. The Alexandrov topology is Hausdorff.

Even stricter assumptions

Stable causality: opening up the light cones "a bit" does not create closed causal curves.

This implies the presence of a global time function.

Causal continuity: the volume of the past and future sets should vary continuously. (Picture on next slide)

Causal simplicity: $J^+(x)$ and $J^-(x)$ are closed.

This means there are no holes in space-time.

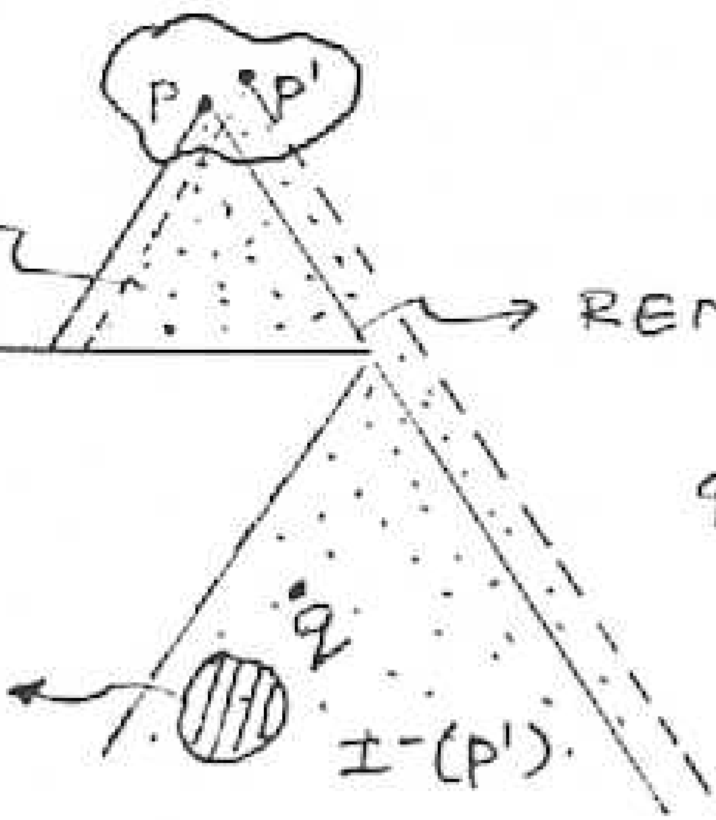
Global hyperbolicity: $J^+(x) \cap J^-(y)$ is compact.

CAUSAL CONTINUITY

VOLUME OF PAST & FUTURE SETS, $I^{\pm}(x)$

SHOULD VARY CONTINUOUSLY WITH x.

$I^-(p)$



REMOVE THIS LINE FROM ^2M.

$q \in I^-(p')$ BUT $q \notin I^-(p)$.

$K \subset M - \overline{I^-(p)}$

$K \not\subset M - \overline{I^-(p')}$

Slide stolen from Sumati Surya.

Topology from Causality

Hawking and King had shown how to reconstruct the topology from $\ll$ and $\leq$.

Malament had shown how to reconstruct the topology from the class of continuous *timelike* curves.

For causally simple space times Keye Martin and I showed that the $\ll$ order is the way-below relation of the causal order.

Thus one can reconstruct $I^{\pm}$ sets from $\leq$.

Which allows one to reconstruct the topology from the order.

Kronheimer-Penrose Axioms

A **causal Space** is a set X equipped with *two* partial orders: $\leq, \ll$ satisfying:

- $\leq$ is a partial order,
- $\ll$ is transitive and irreflexive,
- $\ll \subset \leq$,
- if $x \leq y$ and $y \ll z$ then $x \ll z$,
- if $x \ll y$ and $y \leq z$ then $x \ll z$.

The Horismos

Kronheimer and Penrose use *three* relations: $\leq, \ll$ and $\to$, the last is called the **horismos**.

Definition: $x \to y$ if $x \leq y$ but not $x \ll y$.

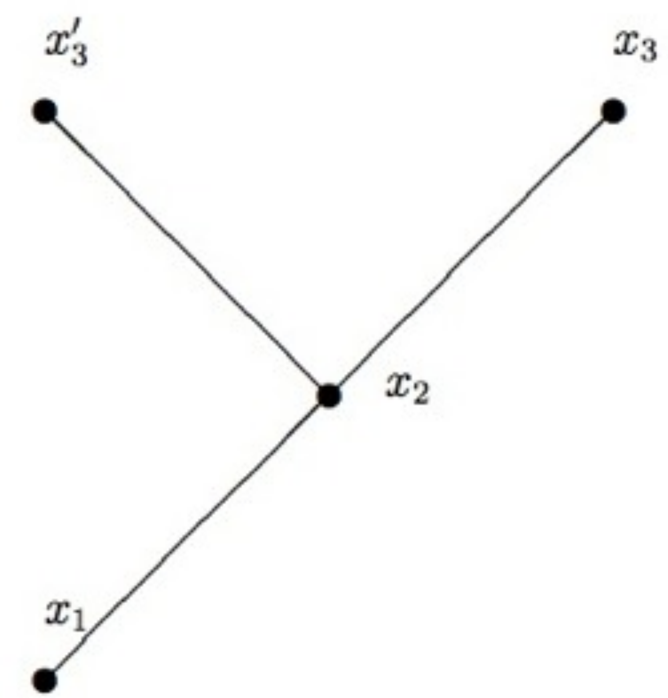
Is there an abstract definition?

> A reflexive binary relation R is called **horismotic** if whenever $(x_i)_{1 \leq i \leq n}$ is a finite sequence with $x_i R x_{i+1}$ for $1 \leq i \leq n$, then for any $1 \leq j \leq k \leq n$:
> (i) $x_1 R x_n$ implies that $x_j R x_k$ and
> (ii) $x_n R x_1$ implies $x_j = x_k$.

The horismos is horismotic

The $\rightarrow$ relation of a causal space is horismotic.

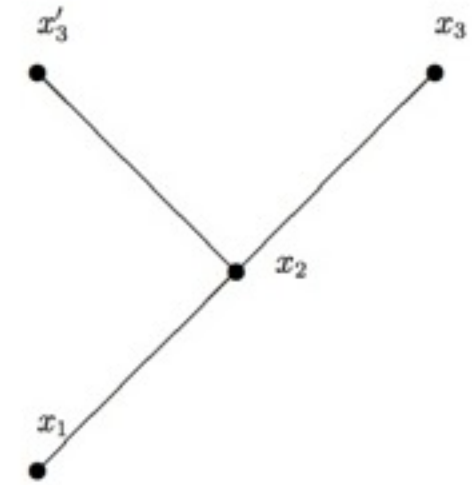
Let $x \leq y \leq z$ in a causal space.
If $x \rightarrow z$ then $x \rightarrow y \rightarrow z$.



Why does that matter?

Imagine a mirror at x_2 which bounces a light ray from x_1 to x_3'.

The path from x_1 to x_3' can be deformed into a timelike path by smoothing the corner.



Anything on the path from x_1 to x_3 has to be $\rightarrow$ related to x_1.

It defines the horismos as the **boundary** of the causal future

without mentioning maximum speeds.

Conclusions

What we do not have in any of the computer science models of causal structure that I have seen: the horismos!

Interesting interplay between causal order and topology.

Recently Martin and I have incorporated the metric using his theory of measurements.

Where does differential structure fit in?

Quantum space times? Perhaps event structure will give us a clue.

Glynn in 1987