Duality in Probabilistic Automata

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- We have discovered an - apparently - new kind of duality for automata.
- Special case of this construction known since 1962 to Brzozowski.
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Joint work with Doina Precup, Joelle Pineau at the RL Lab at McGill and Chris Hundt now working for Google. More recently with Nick Bezhanishvili and Clemens Kupke.

Deterministic Automata

- $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta, \gamma)$: a deterministic finite automaton. S is the set of **states**, $\mathcal{A}$ an **input alphabet** (actions), $\mathcal{O}$ is a set of **observations**.
- $\delta : S \times \mathcal{A} \longrightarrow S$ is the **state transition function.**
- $\gamma : S \longrightarrow \mathbf{2}^{\mathcal{O}}$ or $\gamma : S \times \mathcal{O} \longrightarrow \mathbf{2}$ is a labeling function.
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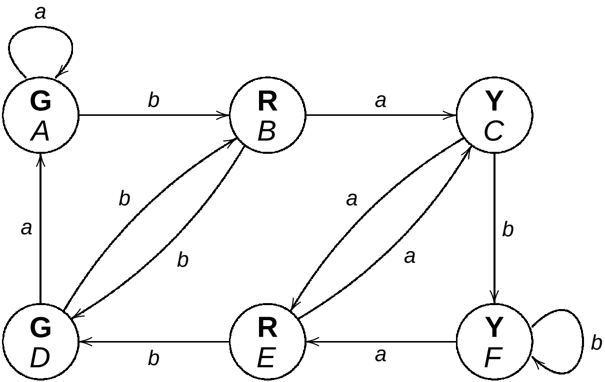
An Example



States are $\{A, B, C, D, E, F\}$ and $\{\mathbf{G}, \mathbf{R}, \mathbf{Y}\}$ are lights.

Testing the Machine

- What can we do with this machine?
- We can ask if *in the present state* the red light is on.
- We can ask whether *after an a-transition* the yellow light is on.
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States Satisfy Tests (Or Not)

- R is satisfied by states $\{B, E\}$
- After a, **red** is on, is satisfied by $\{C, F\}$.
- After ba, **yellow** is on is satisfied by $\{A, D\}$

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A Simple Modal Logic

- Thinking of the elements of $\mathcal{O}$ as formulas we can use them to define a simple modal logic. We define a *formula* φ according to the following grammar:

$$\varphi ::== \omega \in \mathcal{O} \mid (a)\varphi$$

where $a \in \mathcal{A}$.

- We say $s \models \omega$, if $\omega \in \gamma(s)$ (or $\gamma(s, \omega) = T$). We say $s \models (a)\varphi$ if $\delta(s, a) \models \varphi$.
- Now we define $[\![\varphi]\!]_{\mathcal{M}} = \{s \in S | s \models \varphi\}$.

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An Equivalence Relation on Formulas

- We write sa as shorthand for $\delta(s, a)$.
- Define $\sim_{\mathcal{M}}$ between *formulas* as $\varphi \sim_{\mathcal{M}} \psi$ if $[\![\varphi]\!]_{\mathcal{M}} = [\![\psi]\!]_{\mathcal{M}}$.
- Note that this allows us to identify an equivalence class for φ with the set of states $[\![\varphi]\!]_{\mathcal{M}}$ that satisfy φ.
- Note that another way of defining this equivalence relations is

$$\varphi \sim_{\mathcal{M}} \varphi' := \forall s \in S. s \models \varphi \iff s \models \varphi'.$$

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- The formulas **G** and **aG** are only satisfied by A, D. They are thus equivalent.
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A Dual Automaton

- Given a finite automaton $\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta, \gamma)$. Let T be the set of $\sim_{\mathcal{M}}$-equivalence classes of formulas on $\mathcal{M}$.
- We define $\mathcal{M}' = (S', \mathcal{A}, \mathcal{O}', \delta', \gamma')$ as follows:
- $S' = T = \{[\![\varphi]\!]_{\mathcal{M}}\}$
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- $\delta'([\![\varphi]\!]_{\mathcal{M}}, a) = [\![(a)\varphi]\!]_{\mathcal{M}}$
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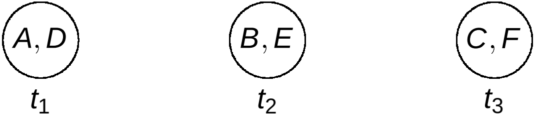
The intuition

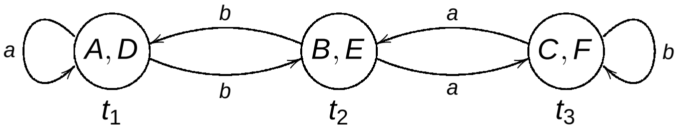
We have interchanged the states and the observations or propositions; more precisely we have interchanged equivalence classes of formulas - based on the observations - with the states. We have made the states of the old machine the observations of the dual machine.

The States of the Dual: Our Example



The states are equivalence classes of formulas but we have labelled them with the *set of states of the original machine* that satisfies the tests.

The Dual Machine in Full: Our Example



The Double Dual

- Now consider $\mathcal{M}'' = (\mathcal{M}')'$, the dual of the dual.
- Its states are equivalence classes of $\mathcal{M}'$-formulas.
- Each such class is identified with a set $[\![\varphi']\!]_{\mathcal{M}'}$ of $\mathcal{M}'$-states by which formulas in that class are satisfied, and
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The Double Dual 2

- Let $\mathcal{M}''$ be the double dual, and for any state $s \in S$ in the original automaton we define

$$Sat(s) = \{[\![\varphi]\!]_{\mathcal{M}} : s \models \varphi\}.$$

- Lemma: For any $s \in S$, $Sat(s)$ is a state in $\mathcal{M}''$.
- In fact *all* the states of the double dual have this form.
- Lemma: Let $s'' = [\![\varphi]\!]_{\mathcal{M}'} \in S''$ be any state in $\mathcal{M}''$. Then $s'' = Sat(s_\varphi)$ for some state $s_\varphi \in S$.
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Minimality Properties

- If $\mathcal{A}$ is reduced then *Sat* is a bijection from S to S''.
- The statement above can be strengthened to show that we actually have an isomorphism of automata.
- If we define a notion of bisimulation we can show that a machine and its double dual are bisimilar.
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The Nondeterministic Case

- Here we consider automata of the type

$$\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta : S \times \mathcal{A} \longrightarrow 2^S, \gamma : S \longrightarrow 2^{\mathcal{O}}).$$

- We use the same formulas but we have a different notion of satisfaction: $Q \subseteq S$

$$Q \models \omega \iff \exists s \in Q : \omega \in \gamma(s)$$

$$Q \models (a)\varphi \iff \delta(Q, a) \models \varphi.$$

- We define an appropriate notion of simulation and prove: $\mathcal{M}$ is simulated by $\mathcal{M}''$.
- The double dual is always deterministic; we have sneaked in the notion of determinization into the satisfaction relation.

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$$Q \models (a)\varphi \iff \delta(Q, a) \models \varphi.$$
- We define an appropriate notion of simulation and prove: $\mathcal{M}$ is simulated by $\mathcal{M}''$.
- The double dual is always deterministic; we have sneaked in the notion of determinization into the satisfaction relation.

Brzozowski's Algorithm 1962

- Take a NFA and just reverse all the transitions and interchange initial and final states.
- Determinize the result.
- Reverse all the transitions again and interchange initial and final states.
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Probabilistic systems

- Everything is discrete.
- Markov Decision Processes aka Labelled Markov Processes:

$$\mathcal{M} = (S, \mathcal{A}, \forall a \in \mathcal{A},\ \tau_a : S \times S \longrightarrow [0, 1]).$$

The τ_a are transition probability functions (matrices).
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Partial Observations

- Partially Observable Markov Decision Processes (POMDPs). We cannot see the entire state but we can see something.
- In process algebra we typically take actions as not always being enabled and we *observe whether actions are accepted or rejected*.
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Automata with State-based Observations

- A **deterministic automaton with stochastic observations** is a quintuple

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- Rather than thinking of propositions and formulas we will think of observations and tests. I will look at state-based notions of observations.
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- We use the same logic as before except that we give a probabilistic semantics and call the formulas “tests.” I write $a.t$ or at rather than $(a)\varphi$.
- Tests define functions from states to $[0, 1]$. If they define the same function they are equivalent.
- The explicit definition of these functions are:

$$[\![o]\!]_{\mathcal{E}}(s) = \gamma(s, o)$$

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- $\mathcal{O}' = S$
- $\gamma'([\![t]\!]_{\mathcal{E}}, s) = [\![t]\!]_{\mathcal{E}}(s)$
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- This machine has deterministic transitions and γ' is just the transpose of γ.

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The Double Dual

- If $\mathcal{E}$ is the primal and $\mathcal{E}'$ is the dual then the states of the double dual, $\mathcal{E}''$ are $\mathcal{E}'$-equivalence classes of tests.
- An "atomic" test is just an observation of $\mathcal{E}'$, which is just a state of $\mathcal{E}$ so it has the form $[\![s]\!]_{\mathcal{E}'}$ for some s.
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Inadequacy of e-tests

- There is a loss of information in the previous construction.
- The double dual behaves just like the primal with respect to "e-tests" but not with respect to more refined kinds of observations.
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More General Tests

- Recall the definition of a POMDP

$$\mathcal{M} = (S, \mathcal{A}, \mathcal{O}, \delta_a : S \times S \longrightarrow [0,1], \gamma_a : S \times \mathcal{O} \longrightarrow [0,1]).$$

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Some Notation

- We need to generalize the transition function to keep track of the final state.

$$\delta_\epsilon(s, s') = \mathbf{1}_{s=s'} \qquad \forall s, s' \in S$$
$$\delta_{a\alpha}(s, s') = \sum_{s''} \delta_a(s, s'')\delta_\alpha(s'', s') \qquad \forall s, s' \in S.$$

- We have written $\mathbf{1}_{s=s'}$ for the indicator function.
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Equivalence on Experiments

- For experiments e_1, e_2, we say
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- The construction of the dual proceeds as before by making equivalence classes of experiments the states of the dual machine and
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- where $S' = \{[e]_{\mathcal{M}}\}$, $\mathcal{O}' = S$
- $\delta'([e]_{\mathcal{M}}, a_0) = [a_0 e]_{\mathcal{M}}$ and
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- We use the e-test construction to go from the dual to the double dual.
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The Main Theorem

- One has to check that everything is well defined.
- The main result is: The probability of a state s in the primal satisfying a experiment e, i.e. $\langle s|e\rangle$ is given by $\langle [s]_{\mathcal{M}'}|[e]_{\mathcal{M}}\rangle = \gamma''([s]_{\mathcal{M}'})|[e]_{\mathcal{M}}\rangle$, where $[s]$ indicates the equivalence class of the e-test on the dual which has s as an observation and an empty sequence of actions.

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AI Motivation

- One can plan when one has the model: value iteration etc., but quite often one does not have the model.
- In the absence of a model, one is forced to learn from data.
- Learning is hopeless when one has no idea what the state space is.
- There should be no such thing as absolute state! State is just a summary of past observations that can be used to make predictions.
- The double dual shows that the state can be regarded as just the summary of the outcomes of experiments.

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What is the right categorical description?

- Is this is any kind of familiar Stone-type duality?
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Automata as Coalgebras

Our automata are coalgebras of the following functor:

$$F(S) = S^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}}, \quad F(f : S \to S') = \lambda(\alpha : \mathcal{A} \to S, \ O \subset \mathcal{O}).(f \circ \alpha, \ O).$$

The category of these coalgebras is called **PODFA.**

Homomorphisms

A homomorphism for these coalgebras is a function $f : S \to S'$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "{(\delta,\ \gamma)}"'] & S' \arrow[d, "{(\delta',\ \gamma')}"] \\ S^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}} \arrow[r, "f^{\mathcal{A}} \times \mathrm{id}"'] & S'^{\mathcal{A}} \times \mathbf{2}^{\mathcal{O}} \end{tikzcd}$$

where $f^{\mathcal{A}}(\alpha) = f \circ \alpha$.

This translates to the following conditions:

$$\forall s \in S, \omega \in \mathcal{O}, \ \omega \in \gamma(s) \iff \omega \in \gamma'(f(s)) \tag{1}$$

and

$$\forall s \in S, a \in \mathcal{A}, \ f(\delta(s,a)) = \delta'(f(s),a). \tag{2}$$

The Dual Category

- The category of **finite boolean algebras with operators** (**FBAO**) has as objects finite boolean algebras B with
- the usual operations $\wedge$, $\neg$ and constants $\top$ and $\bot$ and, in addition,
- together with unary operators (a) and constants $\underline{\omega}$.
- We denote an object by $\mathcal{B} = (B, \{(a)|a \in \mathcal{A}\}, \{\underline{\omega}|\omega \in \mathcal{O}\}, \top, \wedge, \neg)$.

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Morphisms

The morphisms are the usual boolean homomorphisms preserving, in addition, the constants and the unary operators. The following three equations hold:

$$(a)(b_1 \wedge b_2) = (a)b_1 \wedge (a)b_2, \tag{3a}$$

$$(a)\mathsf{T} = \mathsf{T}, \tag{3b}$$

$$\neg(a)\neg b = (a)b. \tag{3c}$$

Duality Theorem

There is a dual equivalence of categories

$$\mathbf{PODFA}^{op} \cong \mathbf{FBAO}.$$

One functor $\mathcal{P}$ is just the contravariant power set functor and the other one $\mathcal{H}$ maps a boolean algebra to its set of atoms.

Minimization?

- Obviously, if we have an equivalence of categories we get the same machine back when we go back and forth.
- So how do we explain the minimization?

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Definable Subsets

Define a logic $\mathcal{L}$ by

$$\phi ::== \mathsf{T} | \bot | \phi_1 \wedge \phi_2 | \neg\phi | (a)\phi | \underline{\omega}$$

and define the **definable subsets** $\mathcal{D}(S)$ of a machine $\mathcal{M} = (S, \delta, \gamma)$ as sets of the form $[\![\phi]\!]$.

- $\mathcal{D}(S)$ is a subobject of $\mathcal{P}(\mathcal{M})$
- in fact it is the *smallest* possible subalgebra and
- any other subalgebra must contain $\mathcal{D}(S)$.

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In Pictures

$$\mathcal{M} \dashrightarrow \mathcal{P}(\mathcal{M})$$

$$\begin{tikzcd} \mathcal{M} \arrow[r, dotted] & \mathcal{P}(\mathcal{M}) \\ & D(S) \arrow[u, hook, "1"'] \end{tikzcd}$$

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\begin{tikzcd}
\mathcal{M} \arrow[r, dotted] \arrow[d, two heads, "2"'] & \mathcal{P}(\mathcal{M}) \\
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The Secret of Minimization

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A Simpler Logic

- Why did the minimization work with just the logic

$$\phi ::== \underline{\omega} | (a)\phi?$$

- With this logic the definable subsets $E(S)$ do not form a boolean algebra
- it is just a "set with operations"
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Deterministic vs Nondeterministic Automata

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- Thus for **deterministic** automata the boolean algebra generated by $E(S)$ is just the same as $D(S)$ so the minimization picture works with boolean algebra generated by $E(S)$.
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Conclusions

- We need to understand the categorical story for nondeterministic automata, probabilistic automata (weighted automata) and perhaps other things as well.
- We are experimenting with these ideas for use in *approximation* in the RL Lab at McGill; joint with Doina Precup and Joelle Pineau and their students.
- Extension to continuous observation and continuous state spaces.
- It is possible to eliminate state completely in favour of histories; when can this representation be compressed and made tractable?
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