FOPPS Lectures: Probabilistic Bisimulation Metrics and Their Applications to Representation Learning

Lecture 1: The logical characterization of bisimulation

Prakash Panangaden
School of Computer Science, McGill University
Montreal Institute of Learning Algorithms
School of Informatics, The University of Edinburgh

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Overview

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- Labelled Markov processes: probabilistic transition systems with continuous state spaces.

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- Diffusion and continuous-time processes [MFPS 2019, 2020]

Collaborators

Giorgio Bacci, Philippe Chaput, Linan Chen, Florence Clerc, Vincent Danos, Josée Desharnais, Abbas Edalat, Norm Ferns, Nathanaël Fijalkow, Robert Furber, Vineet Gupta, Radha Jagadeesan, Bartek Klin, Dexter Kozen, Kim Larsen, François Laviolette, Radu Mardare, Gordon Plotkin and Doina Precup.

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- a set of *labels* or *actions*, L or $\mathcal{A}$ and
- a transition relation $\subseteq S \times \mathcal{A} \times S$, usually written

$$\rightarrow_a \subseteq S \times S.$$

The transitions could be indeterminate (nondeterministic).

Markov Chains

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- The key property is that the transition probability from s to s' only depends on s and s' and not on the past history of how it got there. This is what allows the probabilistic data to be given as a single matrix T.

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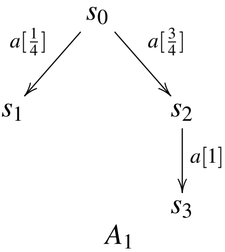
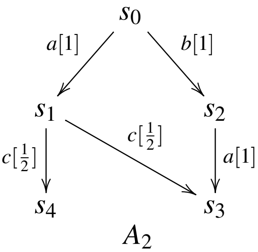
- Just like a labelled transition system with probabilities associated with the transitions.
- $$(S, \mathsf{L}, \forall a \in \mathsf{L}\ T_a : S \times S \longrightarrow [0, 1])$$
- The model is *reactive*: All probabilistic data is *internal* - no probabilities associated with environment behaviour.

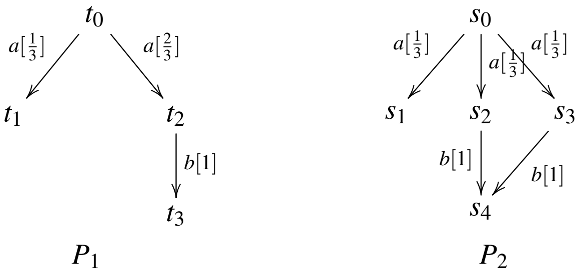
Examples of PTSs





Bisimulation for PTS: Larsen and Skou

- Consider



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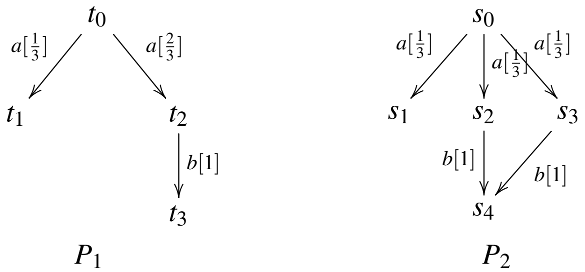

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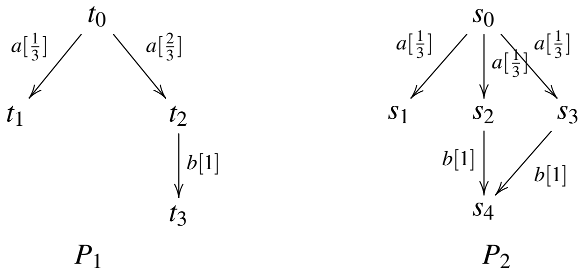

- Should s_0 and t_0 be bisimilar?
- Yes, but we need to add the probabilities.

The Official Definition

- Let $\mathcal{S} = (S, \mathsf{L}, T_a)$ be a PTS. An equivalence relation R on S is a **bisimulation** if whenever sRs', with $s, s' \in S$, we have that for all $a \in \mathcal{A}$ and every R-equivalence class, A, $T_a(s, A) = T_a(s', A)$.

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- The notation $T_a(s, A)$ means "the probability of starting from s and jumping to a state in the set A."
- Two states are bisimilar if there is some bisimulation relation R relating them.

What are labelled Markov processes?

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- All probabilistic data is *internal* - no probabilities associated with environment behaviour.
- We observe the interactions - not the internal states.
- **In general, the state space of a labelled Markov process may be a *continuum*.**

The Need for Measure Theory

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- More precisely, there is no non-trivial translation-invariant measure defined on all the subsets of the reals.

Stochastic Kernels

- A *stochastic kernel* (Markov kernel) is a function $h : S \times \Sigma \longrightarrow [0, 1]$ with (a) $h(s, \cdot) : \Sigma \longrightarrow [0, 1]$ a (sub)probability measure and (b) $h(\cdot, A) : S \longrightarrow [0, 1]$ a measurable function.

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- They are the Kleisli arrows of a monad: the Giry monad.

Formal Definition of LMPs

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- $\forall s : S.\lambda A : \Sigma.\tau_\alpha(s, A)$ is a subprobability measure
  and
  $\forall A : \Sigma.\lambda s : S.\tau_\alpha(s, A)$ is a measurable function.

Probabilistic Bisimulation

Desharnais et al.

Let $\mathcal{S} = (S, i, \Sigma, \tau)$ be a labelled Markov process. An equivalence relation R on S is a **bisimulation** if whenever sRs', with $s, s' \in S$, we have that for all $a \in \mathcal{A}$ and every R-closed measurable set $A \in \Sigma$, $\tau_a(s, A) = \tau_a(s', A)$.

Two states are bisimilar if they are related by a bisimulation relation.

A game for bisimulation

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- We prove that x is bisimilar to y iff Duplicator has a winning strategy starting from (x, y).

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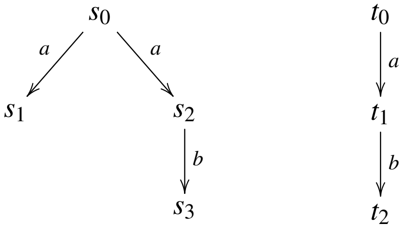
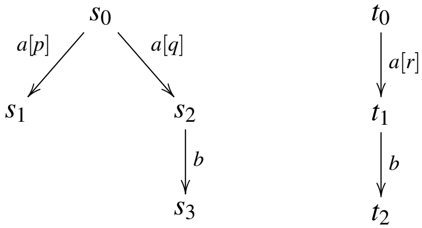
  $$\exists A \in \Sigma.(\forall s' \in A. s' \models \phi) \wedge (\tau_a(s, A) > q).$$
- Two systems are bisimilar iff they obey the same formulas of $\mathcal{L}$. [DEP 1998 LICS, I and C 2002]

That cannot be right?



Two processes that cannot be distinguished without negation.
The formula that distinguishes them is $\langle a \rangle (\neg \langle b \rangle \top)$.

But it is!

- 

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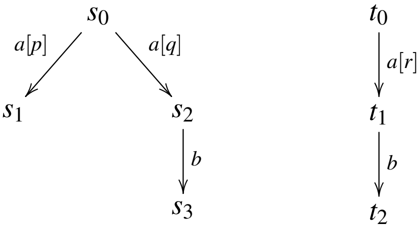
- 

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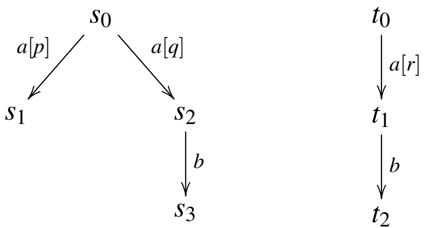
- 

  We add probabilities to the transitions.
- If $p+q<r$ or $p+q>r$ we can easily distinguish them.
- If $p+q=r$ and $p>0$ then $q<r$ so $\langle a\rangle r\langle b\rangle 1\top$ distinguishes them.

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- Use Dynkin's lemma to show that we get a well defined measure on the σ-algebra generated by such sets and the above equality holds.
- Use special properties of analytic spaces to show that this σ-algebra is the same as the original σ-algebra.

Simulation

Let $\mathcal{S} = (S, \Sigma, \tau)$ be a labelled Markov process. A preorder R on S is a **simulation** if whenever sRs', we have that for all $a \in \mathcal{A}$ and every R-closed measurable set $A \in \Sigma$, $\tau_a(s, A) \leq \tau_a(s', A)$. We say s is simulated by s' if sRs' for some simulation relation R.

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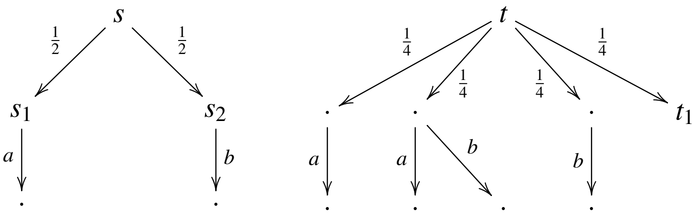
- The logic used in the characterization has no negation, not even a limited negative construct.
- One can show that if s simulates s' then s satisfies all the formulas of $\mathcal{L}$ that s' satisfies.
- What about the converse?

Counter example!

In the following picture, t satisfies all formulas of $\mathcal{L}$ that s satisfies but t does not simulate s.



All transitions from s and t are labelled by a.

Counter example (contd.)

- A formula of $\mathcal{L}$ that is satisfied by t but not by s.

$$\langle a\rangle_0(\langle a\rangle_0\mathsf{T} \wedge \langle b\rangle_0\mathsf{T}).$$

Counter example (contd.)

- A formula of $\mathcal{L}$ that is satisfied by t but not by s.

$$\langle a \rangle_0(\langle a \rangle_0 \mathsf{T} \wedge \langle b \rangle_0 \mathsf{T}).$$

- A formula with disjunction that is satisfied by s but not by t:

$$\langle a \rangle_{\frac{3}{4}}(\langle a \rangle_0 \mathsf{T} \vee \langle b \rangle_0 \mathsf{T}).$$

A logical characterization for simulation

- The logic $\mathcal{L}$ does **not** characterize simulation. One needs disjunction.

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  $$s_1 \models \phi \Rightarrow s_2 \models \phi.$$
- The original proof uses domain theory and approximation.
- New development (2017 ICALP) we can prove logical characterization for simulation and bisimulation in almost the same way.

Digression on Analytic Spaces

- An analytic set A is the image of a Polish space X (or a Borel subset of X) under a continuous (or measurable) function $f : X \to Y$, where Y is Polish. If (S, Σ) is a measurable space where S is an analytic set in some ambient topological space and Σ is the Borel σ-algebra on S.

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- Analytic sets do not form a σ-algebra but they are in the completion of the Borel algebra under **any** measure. [Universally measurable.]

Amazing Facts about Analytic Spaces

- Given A an analytic space and $\sim$ an equivalence relation such that there is a *countable* family of real-valued measurable functions $f_i : S \longrightarrow \mathbf{R}$ such that

$$\forall s, s' \in S . s \sim s' \iff \forall f_i . f_i(s) = f_i(s')$$

then the quotient space (Q, Ω) - where $Q = S/\sim$ and Ω is the finest σ-algebra making the canonical surjection $q : S \longrightarrow Q$ measurable - is also analytic.

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  then the quotient space (Q, Ω) - where $Q = S/\sim$ and Ω is the finest σ-algebra making the canonical surjection $q : S \longrightarrow Q$ measurable - is also analytic.
- If an analytic space (S, Σ) has a sub-σ-algebra Σ_0 of Σ which separates points and is countably generated then Σ_0 is Σ! The Unique Structure Theorem (UST).

Some more measure theory

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Some more measure theory

- A π-system is a family of sets closed under finite intersections.
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- Corollary: If two measures agree on the sets of a π-system then they agree on the generated σ-algebra.

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- Let $\delta = \tau_a(x, \cdot)$ and $\gamma = \tau_a(y, \cdot)$.
- If $\delta(S) > \gamma(S)$ then choose *rational* q such that $\delta(S) > q > \gamma(S)$. Now $x \models \langle a \rangle_q \top$ and $y \not\models \langle a \rangle_q \top$.

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- $x \preceq y$ iff Duplicator has a winning strategy starting from x, y.

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- With these in place the proof of the logical characterization of simulation follows the same pattern.

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- We heavily use topological ideas in this proof.

What is next?

Metrics!