Alan Turing's Contributions to the Sciences

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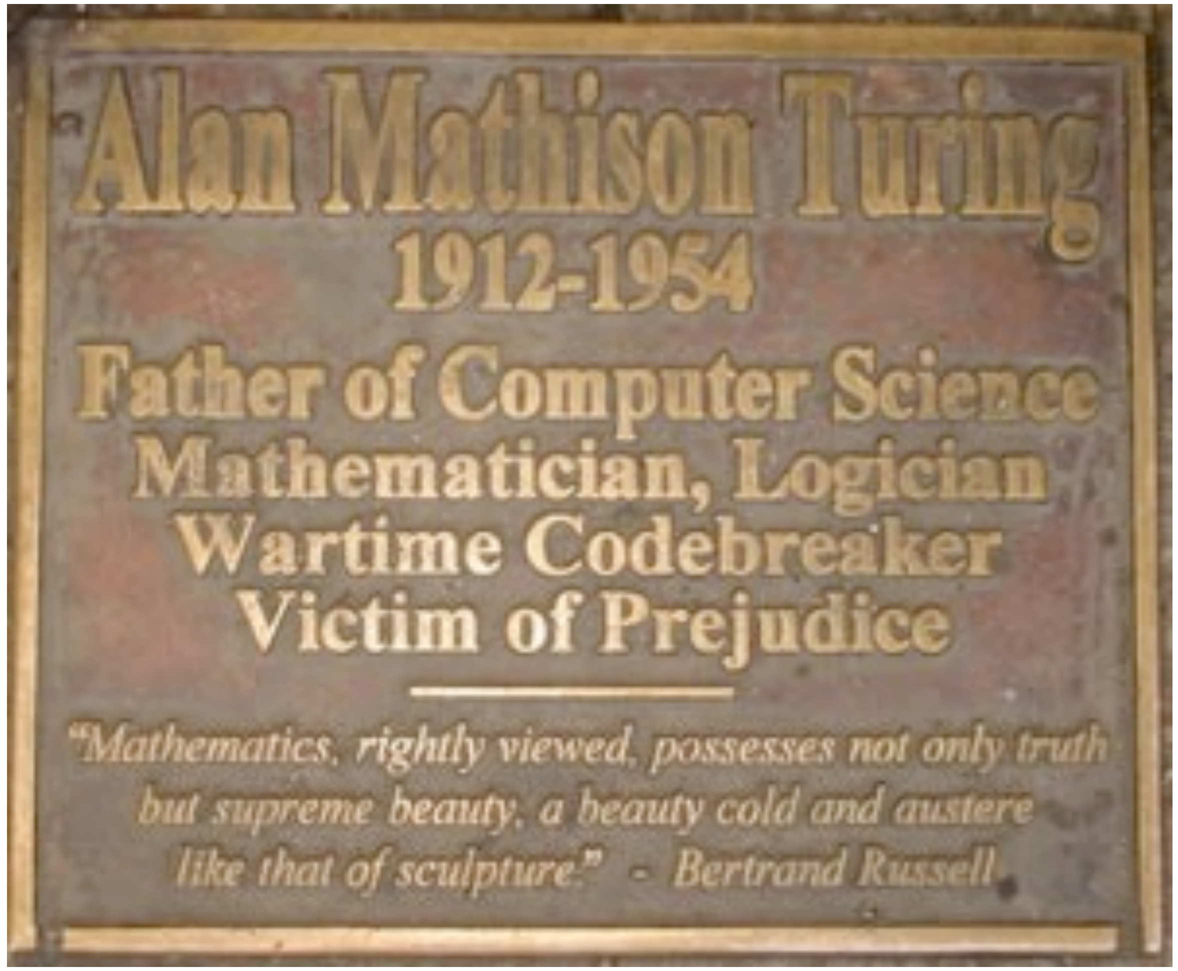
Alan Mathison Turing
1912-1954
Father of Computer Science
Mathematician, Logician
Wartime Codebreaker
Victim of Prejudice
"Mathematics, rightly viewed, possesses not only truth
but supreme beauty, a beauty cold and austere
like that of sculpture." - Bertrand Russell

Who is Alan Turing?

- Logician
- Mathematician
- Cryptanalyst
- Computer architect
- Mathematical biologist
- Inventor of computer science

More precisely

- Gave a compelling notion of effective computability.
- Showed the equivalence with Church's Lambda Calculus.
- Formulated the notion of a universal machine; thus inventing the concept of software.
- Demonstrated the existence of unsolvable problems (slightly later than A. Church).

and more!

- Formulated the Turing test and initiated AI.
- Designed the ACE stored-program computer.
- Pioneered numerical analysis.
- Formulated the concept of computable real numbers.
- Proved normalization of simply-typed lambda calculus.

yet more!!

- Developed the theory of pattern formation via reaction-diffusion equations and laid the foundations for the chemical basis of morphogenesis.
- Introduced the systematic study of error propagation in numerical computation and invented the concept of condition number.

Less well known

- First paper on proving program correct.
- Using computers to play games.
- Significant contributions to type theory.
- Finite approximations to Lie groups.
- Computing the Riemann zeta function.

My focus today: normal numbers

- Borel defined a normal number in 1900 and showed that with probability 1 a “random number” is normal.
- But he could not explicitly exhibit one!
- He posed it as a major problem.
- Turing gave an algorithm to compute normal numbers.

Normal numbers as random numbers

- In the decimal expansion of a "random" real number the digits should appear equally often.
- Is .012345678901234567890123456789... a random number? Very predictable pattern.
- We want every finite sequence of digits to occur as often as any other sequence of the same length.

Precise definitions

Consider numbers $\alpha \in (0,1)$ represented to base b. We write

$$\alpha = \sum_{n=1}^{\infty} a_n b^{-n}$$

where a_n are integers in $\{0, \ldots, b-1\}$ and $a_n < b-1$ infinitely often in order to ensure that rational numbers have a unique representation.

Let w be a word of length k formed from the digits $\{0, \ldots, b-1\}$. We write $\alpha(n)$ for the digit at position n of the base-b expansion of α. We write $S(w, N) = |\{n : \alpha(n)\alpha(n+1)\ldots\alpha(n+k-1) = w\}|$, where $n+k-1 \leq N$. $S(w, N)$ is the number of times the word w occurs in the first N digits of the base-b expansion of α.

We say α is **normal to base** b if for every word w we have

$$\lim_{N \to \infty} S(w, N)/N = b^{-k}.$$

Clearly, rational numbers cannot be normal to any base.

A number is **absolutely normal** if it is normal to every base.

A number is **simply normal** to base b if each digit has asymptotic frequency $\frac{1}{b}$ in its base b expansion. A number is absolutely normal if it is simply normal to every base.

What we do and don't know

Are there numbers that are normal to one base
but not to another? Yes!

Examples? Not a single one known!

Do we know any "natural" examples of normal numbers?
Like $e, \pi, \sqrt{2}, \zeta(3)$. No!

In fact we do not know if any of these are normal to any base!!

Conjecture: all algebraic irrational numbers are normal.

Fact: Champernowne's number
$\cdot 0123456789101112131415161718192021 22\ldots$

is normal to base 10 but not known to be normal
to any other base.

Sierpinski's contribution

In 1916 Sierpinski gave a "method" for producing an explicit example of a normal number.

He defines, for every $\varepsilon > 0$, a countable family of intervals, $\Delta(\varepsilon)$, and shows that any number not absolutely normal is in $\Delta(\varepsilon)$ and that $|\Delta(\varepsilon)| < \varepsilon$.

This has a "constructive" character and one view this as an explicit description of a normal number.

In 2002 Veronica Becher and Santiago Figueira gave an algorithm to produce a *computable* absolutely normal number.

Turing's contribution

Theorem 1 (Turing's first theorem). *There is a computable function* $c : \mathbb{N} \times \mathbb{N} \to \mathcal{P}((0,1))$ *such that*

1. *$c(k,n)$ is a finite union of intervals with rational endpoints;*
2. *$c(k,n+1) \subseteq c(k,n)$;*
3. *$\mu\left(c(k,n)\right) > 1 - 1/k$.*

and for each k, $E(k) = \bigcap_n c(k,n)$ has measure $1 - 1/k$ and consists entirely of absolutely normal reals.

A number α is in $c(k,n)$ if the frequency of *every word* of length up to L that occurs in the first N digits of the base-b expansion occurs with the expected frequency $\pm N\varepsilon$ for *every* base b up to B where N, B, L, ε are effectively given in terms of n, k.

The proof of all these claims depends on a lemma not proved by Turing. Becher et al. (2007) proved a weaker version of the lemma, which sufficed to establish all the properties of $c(k, n)$.

This theorem shows that the non-normal numbers are contained in an effectively given null set.

Turing also derives from this another theorem which defines an algorithm to produce the binary expansion of an absolutely normal number.

Theorem 2 (Turing's second theorem)**.** *There is an algorithm that, given $k \in \mathbb{N}$ and an infinite sequence $\theta \in \{0,1\}^\infty$, produces an absolutely normal real number $\alpha \in (0,1)$ in the scale of 2. For a fixed k these numbers α form a set of Lebesgue measure at least $1 - 2/k$, and so that the first n digits of θ determine α to within 2^{-n}.*

Turing's proof contains some gaps and needed some bug fixes as well. These were provided by Becher et al.

They also showed that producing n bits requires time $O(2^{2^n})$, contrary to Turing's claim that it required a single exponential.

In 2007 Elvira Mayordomo gave an $O(n \log n)$ algorithm to compute absolutely normal numbers.

Conclusions

Turing's paper on normal numbers was not published until his collected works appeared in 1992.

It was a visionary paper opening the way to the modern theory of algorithmic randomness.

This theory has led to Martin-Löf's characterization of a random sequence (1966).

Many modern developments: algorithmic information theory, fractal dimension of finite state recognizers, computability and randomness, ...