MAT 9455 Algebraic Combinatorics and Coinvariant Spaces
Scribe: Matt Drescher
Instructor: Prof Bergeron **winter, 2006**

Has applications to :
Algebraic Geometry
Representation Theory
Combinatorics
Symmetric Functions
physical statistics

S_n: the symmetric group of n elements.
$Q[x]$: The ring of polynomials in n variables
$x := x_1 x_2 ... x_n$
$x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$ where $\alpha := (\alpha_1, \alpha_2, ..., \alpha_n) \in N^n$

$deg(x^\alpha) = |\alpha| := \alpha_1 + \alpha_1 + ... + \alpha_n$

$$\pi_d : Q[x] \to Q[x]$$

$$\pi_d(x^a) = x^a \ if \ |a| = d \ \ , \ and \ 0 \ otherwise$$

Definition 1 *A subset $I \subset Q[x]$ is an **ideal** if it satisfies:*
(i) $0 \in I$
(ii) If $a, b \in I$, then $a + b \in I$
(iii) if $a \in I$ and $h \in Q[x]$ then $ah \in I$

if I is an ideal then there exists a homomorphism $Q[x] \to \frac{Q[x]}{I}$.
notice $\pi_d(I) \subseteq I$

We have
$R := Q[x]$ $R = \bigoplus_{d=1}^{\infty} R_d \ with \ R_d := \pi_d(R)$
and in general if V is a subspace of R, it is said to be *homogenious* if
$V \cong \bigoplus_{d \geq 0} V_d \ with \ V_d := \pi_d(R)$

The *Hilbert series* we define as

$$H_V(q) := \sum_{d \geq 0} dim(V_d) q^d$$

for R we have:

$$dim(R_d) = \binom{n - 1 + d}{d}$$

so

$$H_R(q) = \sum_{d \geq 0} \binom{n-1+d}{d} q^d$$
$$= \frac{1}{1-q}^n$$

(by generalized binomial theorem)

$$I_{S_n} := (h_1(x), h_2(x), ..., h_n(x))$$

with

$$h_d(x) = \sum_{|a|=d} x^a$$

$(|a| = a_1 + a_2 + ... + a_n)$.

$h_d(x)$ is a polynomial symmetric invariant for S_n.
*****

$$S_n = \frac{Q[x]}{I_{S_n}}$$

*****

Properties:
(1) C_n is ring of cohomology of the variety of flags(??)
(2) $dim(C_n) = n!$
(3) $H_{S_s}(q) = (1+q)(1+q+q^2)...(1+q+...+q^{n-1})$.

$$H_{S_n}(q) = \sum_{\sigma \in S_n} q^{inv(\sigma)}$$

$(-1)^{\#\ of crossings} = (\sigma)$
minimal number of crossings = $inv(\sigma)$.

$dim(\pi_d(S_n))$ = number of permutations which have d inversions.

0.1 Grobner Bases (of an ideal)

$$(I_{S_n})\ I_n = (h_1(x), h_2(x), ..., h_n(x))$$

look up some examples.

0.2 Stage one choose an order monomial

(a)
Order the variables such that

$$x_1 < x_2 < x_3 < ... < x_n$$

(b)
order the monomials such that

$$x^\alpha < x^\beta \Rightarrow x^\alpha x^d < x^\beta x^\alpha$$

1 Grobner Bases

reference: **"'Using Algebraic geometry"'** Cox, little, O'shea

Bon ordre sur les Monomes.

$$x^a > x^b \quad x_1 > x_2 >, ..., > x_m$$

Compatable with multiplication.

Ex. $x^a > x^b$ soit $deg(x^a) > deg(x^b$ ou $\frac{x^a}{x^b}$ lexposant non-nul de la variable la plus grande est positif. ???? 2 variables $x^a y^b$ if I is an ideal the polynomials have 2 variables.

$$I = (f_1(x,y), f_2(x,y), ..., f_m(x,y))$$

$$D(f) = \ Monomial\ Dominantes\ the\ f(x,y)$$

$$f(x,y) = cx^a y^b + ..._{more\ small\ monomials}$$

$$D(f) = x^a y^b$$

If $f(x,y) \in I$, Acons?? $x^c y^d f(x,y) \in I.$ if $x^a y^b \in D(I)$ acors $x^{a+c} y^{b+d} \in D(I)$

Base De Grobner(minimale, reduite)

$$(g_1, g_2, ..., g_k) = I$$

(1) with each monomial in $D(I)$ is divisible by lun $D(g_l)$
(2) $D(g_i)$ do not divide aucon terms of g_l pour $i \neq l$

(A) for an order monomial gives this base is unique
(B) Effectivement calculable.

connection with stair case picture: Base vectorielle

$$f(x,y) \in_{Attention} \frac{Q[x,y]}{I}$$

$$B = \{x^a | x^a is\ divisible\ by\ none\ of D(g_i)\}$$

B form a base(vectorielle) of $\frac{Q[x]}{I}$

$$dim(\frac{Q[x]}{I}) = \#B$$

$$\sum_{b \in B} q^{deg(b)} = H_{\frac{Q[x]}{I}}(q)$$

1.1 L'exemple:

$$Q[x_1, ..., x_n]/(h_1, ..., h_n)$$

ou

$$h_k(x_1, ..., x_n) = \sum_{|a|=k} x^a \quad a = (a_1, ..., a_n), |a| = \sum a_i$$

$$h_1 = x_1 + x_2 + ... + x_n$$

$$h_2 = x_1^2 + x_1x_2 + ... + x_1x_n + [x_2^2 + x_2x_3 + ... + x_2x_n + ... + x_{n-1}x_n$$

$$= x_1h_1(x_1, ..., x_n) + h_2(x_2, ..., x_n)$$

$$h_3 = x_1h_2(x_1, ..., x_n) + h_3(x_2, ..., x_n)$$

$$h_k(y_1, ..., y_l) = y_1h_{k-1}(y_1, ..., y_l) + h_k(y_2, ..., y_l)$$

$$(h_k(y_2, ..., y_l) = h_k(y_1, ..., y_l) - y_1h_k(y_1, ..., y_l)$$

On A Donc

$$h_k(x_k, ..., x_n) \in I_n$$

(Proof by recurrence)

$$D(h_k(x_k, ..., x_n) = x_k^k$$

$$(h_1(x), ..., h_n(x)) = (h_1(x_1, ..., x_n), h_2(x_2, ..., x_n), ..., h_n(x_n)$$

Prop: Les $h_k(x_k, ..., x_n)$, $1 \leq k \leq n$, formeut a Grobner basis of the ideal I_n. Termes Dominants corresponding is

$$x_1 \ \ x_2^2 \ , \ \ x_3^3 \ , \ \ x_4^4, ... \ .. \ x_n^n$$

$$B = \{x_1^{e_1}x_1^{e_1} \cdots x_n^{e_n} | 0 \leq e_i < i\}$$

Forms a base $\frac{Q[x]}{(h_1, ..., h_n)}$

$$\#B = n!$$

$n = 3$ variables x, y, z
$B = \{1, x_2, x_3, x_2x_3, x_3^2, x_2x_3^2\}$

$$A := \frac{Q[x]}{I_n}$$

$$H_A(q) = 1 + q + q + q^2 + q^2 + q^3 = 1 \cdot (1+q)(1+q+q^2)$$

$$H_A(q) = (1+q)(1+q+q^2)....(1+q+...+q^{n-1})$$

$$Q[x] = I_n \bigoplus V$$

Propuit scavaire on $Q[x]$. $d = differential\ operator$

$$< 3x^2y + xy^2, 2x^3 + x^2y - xy^2 >= 3(\tfrac{d^2}{dx^2}\tfrac{d}{dy} + \tfrac{d}{dx}\tfrac{d^2}{dy^2})(2x^3 + x^2y + xy^2)|_{x=0,y=0}$$

$$< x^a, x^b >= a!\ if\ a = b\ 0\ otherwise$$

$$a! = a_1!a_2! \cdots a_n!$$

Propwery of (inverse system) $I^\perp$(complement orthogonal)D'ideaux I:

$$I^\perp := \{g(x)|[f(dx)g(x)|_{x=0} = 0]_{Terme\ constant}\ for\ all\ f(x) \in I$$

$$I^\perp = \{g(x)|[f(dx)g(x) = 0]_{plus\ fort!!}\ for\ all\ f(x) \in I\}$$

$$= \{g(x)|f_1(dx)g(x) = 0, f_2(dx)g(x) = 0, ..., f_m(dx)g(x) = 0\}$$

$$I = (f_1, f_2, ..., f_m)$$

Dans L'exemple: $g(x) \in I_n^\perp$ ssi
$(dx_1 + dx_2 + ... + dx_m)g(x) = 0$
****
$(d^2x_1 + d^2x_2 + ... + d^2x_m)g(x) = 0$
****
.... $(d^mx_1 + d^mx_2 + ... + d^mx_m)g(x) = 0$

Traditional
$\nabla g(x) = 0$, $g(x)$ ext harmonic $\nabla = (d^2x_1 + d^2x_2 + ... + d^2x_m)g(x)$ Laplacian

$g(x) \in I^\perp, f(x) \in I$, $dx^a = dx_1^{a_1}dx_2^{a_2}....dx_m^{a_m}\ x^af(x) \in I$

[A] $f(dx)g(x) = cx^a + ...$

$$0 = dx^af(dx)g(x)|_{x=0} = ca! \Rightarrow c = 0$$

$(h_1, h_2, ..., h_m) = (k_1, k_2, ..., k_m)$ ou $k_k = x_1^k + ... + x_n^k$

$$h_k = \sum c_d k_1^{a_1} k_2^{a_2} \cdots k_n^{a_n}$$
$$k_l = \sum d_a h_1^{a_1} h_2^{a_2} \cdots h_n^{a_n}$$

$$*****$$

$$\sum_{l\geq 1} h_l t^l = e^{\sum_{k\geq 1} k_k \frac{t^k}{k}}$$

$$*****$$

$$dim(I_n^{\perp}) = n!$$

$$H_{I_n^{\perp}}(q) = 1 + (n-1)q + ... + q^{\binom{n}{2}}$$
$$(1+q)(1+q+q^2)\cdots(1+q+...+q^{n-1})$$

2^l observation si $g(x) \in I_n^{\perp}$ Alors $dx^a g(x) \in I_n^{\perp}$

$$f(dx)g(x) = 0 \ \forall f \in I$$

$$f(dx)dx^a g(x) = dx^a[f(dx)g(x)]_{=0} = 0$$

Determinant De Vandermonde

$$\triangle_n(x) := \prod_{i<j}(x_i - x_j)$$

Proposition 1 $\triangle_n(x) \in I_n^{\perp}$

Proposition 2 $I_n^{\perp} = \{g(dx)\triangle_n(x) | g(x) \in Q[x]\}$

Proposition 3 *a Basis of* $I_n^{\perp}$ *is donnee by* $\{dx^e \triangle_n | e = (e_1, ..., e_n), 0 \leq e_i < i\}$

Preuve Que

$\{dx^e \triangle_m(x)\}_e$ lineairement independant

(avec prop 1 $\Rightarrow dim(I_m^{\perp}) \geq n!$)

another order on the monomials $x_m > x_{m-1} > ... > x_2 > x_1$

Will have the term perminant of$\triangle_n(x)$ est.

$$\delta = (0, 1, ..., n-1) \ \ \triangle_m(x) = x_m^{m-1} x_{m-1}^{m-2} \cdots x_2 \pm \cdots$$

$$e = (e_1, e_2, ..., e_m)$$
$$dx^e = ctex^{\delta - a} \pm ...$$

$$prop1 \Rightarrow prop2 \Rightarrow prop3$$

all decoule de prop 1 $+dim(I_m^{\perp}) \leq n!$

2 Espace Coinvariant De S_n

$\frac{Q[x]}{I_n}$ I_n Ideal Engendre' par les polynomes invariants(saus terme constant) $f(0) = 0$, $f(\sigma(x)) = f(x) \forall \sigma \in S_n$

$$\frac{Q[x]}{I_n} \cong I_n^{\perp}$$

$$< f(x), g(x) > := f(dx)g(x)|_{x=0}$$

On veut voir que

$$\{f(dx)\triangle_n(x)|f(x) \in Q[x]\} \subseteq I_n^{\perp}\}$$

il suffit de verifier que

$$\triangle_n(x) = \prod_{i>j}(x_i - x_j) \in^? I_n^{\perp}$$

ssi

$$f(dx)\triangle_n(x) = 0$$

Pour tout $f(x)$ symetrique.

$$\sigma f(dx)\triangle_n(x) = signe(\sigma)f(dx)\triangle_n(x)$$

$$deg(f(dx)\triangle_n(x)) < deg(\triangle_n(x))$$

$\triangle_n(x)$ est le (non nul) plus petit polynomial *antisymetrique*

$$\sigma \cdot \triangle_n(x) = signe(\sigma) \cdot \triangle_n(x)$$

$$\triangle_n(x) = det(x_i^{n-j})_{1<i,j \leq n} = det(M)$$

$$M := \begin{array}{ll} x_n^{n-1} & \text{..ax}_n^{n-2} x_n^{1} \\ x_{n-1}^{n-1} & \text{..ax}_{n-1}^{n-2} x_{n-1}^{1} \\ \ldots & \\ x_1^{n-1} & \text{..ax}_1^{n-2} x_1^{1} \end{array}$$

si $g(x_1, x_2, ..., x_n)$ est antisymmetrique, alors $g(x)$ se divise par $\triangle_n(x)$(saus reste)

Proof.

$$\sigma \cdot g(x) = signe(\sigma)g(x)$$

$$(i, j)g(x) = -g(x)$$

$$g(x_1, ..., x_i, ..., x_j, ..., x_n)_{i \leftrightarrow j} = 0$$

$(x_i - x_j)$ divise $g(x)$ $\forall i \neq j$ ∎

On a vu que

$$dim(\{f(dx)\triangle_n(x)|f(x) \in Q[x]\}) \geq n!$$

Plisque la famile

$$dx^e \triangle_n(x)$$

$$e := (e_1, ..., e_n) \in N^n \ 0 \leq e_i < i$$

Est linearement ind(linear independant).

$$n! \leq dim(I_n^{\perp}) = dim(\frac{Q[x]}{I_n})$$

Proposition 4 $dim(\frac{Q[x]}{I_n}) = n!$

Proof. (Geometrie Algebrique 101.) (Cas S_n)
Fait # 1:

$$\frac{Q[x]}{I_n} \cong \frac{Q[x]}{GR(I)}$$

(espace vectoriel) I Homogene

$$GR(I) = (m(f(x))|f(x) \in I)$$

$$m(f(x)) := Composant\ Homogene\ De\ Degue\ maximal\ de\ f(x)$$

$$m(x^2 + xy + y^2 - 2x + 3) = x^2 + xy + y^2$$

Base de Grobner avec un ordre $backwards \in$ symbol $x^a > x^b Si$deg(x^a) $> deg(x^b)$

$$\Rightarrow D(f(x)) = D(m(f(x)))$$

Donc

$$(g_1, ..., g_k)$$

ssi

$$m(g_1), ..., m(g_k))$$

Base de grobner de $GR(I)$
Fait # 2
Soit $c = (c_1, ..., c_n) \in Q^n$ Regulier ,i.e:$|\{\sigma c|\sigma \in S_n\}| = n!$ this can be generalized to any group G and we have $|G|$ rather then $n!$.

$$I_c = \{f(x) \in Q[x]|f(\sigma\ c) = 0\ pour\ tout\ sigma \in S_n$$

$$dim(\frac{Q[x]}{I_c}) = \#\{\sigma \cdot c|\sigma \in S_n\} = n!$$

$$\frac{Q[x]}{I_c}$$

$$f(x) \equiv_{I_c} g(x)\ si\ f(x) - g(x) \in I_c$$

$$f(\sigma \cdot c) = g(\sigma \cdot c)$$

pour tout $\sigma \in S_n$
Une Base (vectorielle) de $\frac{Q[x]}{I_c}$ peut donc etre la suivaite: **il existe un polynome** $\alpha_c(x)$

$$\alpha_c(\sigma \cdot c) = 1\ si\ \sigma = Id\ \ sinon$$

$$\alpha_{\sigma\cdot c}(x) := \alpha_c(\sigma^{-1}x)$$

$$\alpha_{\sigma\cdot c}(\tau\cdot\sigma) = 1 \; si \; \tau = \sigma \;\; 0 \; sinon$$

$$*****$$

$$\{\alpha_{\sigma c}(x) | \sigma \in S_n\}$$

$$*****$$

$$f(x) \equiv_{I_c} \sum_{\sigma \in S_n} f(\sigma c)\alpha_{\sigma c}(x)$$

$$\alpha(x) \; Polynome \; interpolateur \; (De \; Lagrange \;)$$

$$\alpha_c(x) :=$$

$$dim(Q[x]/I_n) = |\{\sigma c | \sigma \in S_n\}| = n!$$

$$\{f(x) - f(c) | f(x) \; symetrique\} \subseteq I_c$$

$$\{f(x) | f(x) \; symetrique \; et \; homogene\} \subseteq GR(I_c)$$

$$f(0) = 0$$

$$I_n =^? GR(I_c)$$

$$f(\sigma x) - f(\sigma c) = f(x) - f(c)$$

$$f(x) - f(c)|_{x=c} = 0$$

$$dim(Q[x]/I_n) \leq^? n!$$

$$dim(\frac{Q[x]}{GR(I_c)}) = n!$$

$$*****$$

$$dim(\frac{Q[x]}{I_n}) = n!$$

$$*****$$

$$H_{S_n}(q) = \prod_{k=1}^{n} \frac{1-q^k}{1-q}$$

$$L_n := \frac{Q[x]}{I_n}$$

$$H_{Q[x]}(q) = \frac{1}{(1-q)^n}$$

$$Q[x]^{S_n} = \{f(x) | \sigma \cdot f(x) = f(x)\ \forall \sigma \in S_n\}$$

$Q[x]_d^{S_n}$: Partage(partition??) de d en

$$Q[x] = \bigoplus_{d \geq 0} Q[x]_d$$

$$Q[x]_d \textit{ composante homogene de deore } d$$

$$Q[x]^{S_n} = \bigoplus_{d \geq 0} Q[x]_d^{S_n}$$

$Q_d^{S_n}$ Polynome symi homogene de degrue d
$dim(R_d^{S_n})$ =number of partages de d eu au plus n parts.

Consequences

open problem can we find a natural permutation σ so that we have $n!$ –see ameerahrahrahrah
(1) Tout polynome admet une unique decomposition

$$p(x) = \sum_e x^e f_e(x)$$

$$\{x^e\}_e \textit{ Base de } L_n$$

$$f_e(x) \in Q[x]^{S_n}$$

(2)

$$Q[x] \cong Q[x]^{S_n} \otimes_Q L_n \cong Q[x]^{S_n} \otimes_Q I_n^{\perp}$$

Espace des invariants–$Q[x]^{S_n}$
L'Espace Coinvariant – L_n

$$p(x) = \sum_{\sigma} dx^{e(\sigma)} \triangle_n(x) f_\sigma(x)$$

Autre base $f_\sigma(x) \in Q[x]^{S_n}$ de $I_n^{\perp}$

(3)
$Q[x]$ est un $Q[x]^{S_n}$- module **libre**

$$2 \Leftrightarrow H_{Q[x]}(q) = H_{Q[x]^{S_n}}(q) \cdot H_{L_n}(q)$$

Cas S_n

Si G un groupe fini de matrices $n \times n$

$$g \in G \quad gx$$
$$= (g_{11}x_1 + g_{12}x_2 + ... + g_{1n}x_n, g_{21}x_1 + ...g_{2n}x_n, ..., g_{n1}x_1 + ... + g_{mn}x_n)$$

$$*****$$
$$gp(x) = p(gx)$$
$$*****$$

Action De G sur $Q[x] = R$

$R^G = \{f(x)|gf(x) = f(x)\forall g \in G\}$- G invariants

$$I_G := (f(x)|f(x) \in R^G, f(0, ..., 0) = 0)$$
$$L_G := \frac{Q[x]}{I_G}$$

Action de G preserve le degre des polynomes.

Theorem 1 *(Chevaley, Sheppard-Todd, Borel...)*
$R \cong R^G \otimes L_G$ *si et seulement si* G *est un group eugendre pac des reflections.*

$$H_{R^G}(q) = \prod_i \frac{1}{1 - q^{d_i}}$$

$$H_L(q) = \prod_i \frac{1 - q^{d_i}}{1 - q}$$
$$|G| = d_1 d_2 \cdots d_n$$
$$I_G^{\perp} = \{f(dx)\triangle_G(x)|f(x) \in Q[x]\}$$

3 Groupes fini Engendres Paz Des Reflexions

V esp. Vectoriel de dimension $n \cong R^n$ plus Produit Scalaire. H hyperplane $dim(H) = n-1$, $v_2, ..., v_n$ Base De H $v_1 \perp v_i$, $i \geq 2$
Reflection Dans H :

$$S_H(v_i) := -v_i \ si \ i = 1, \ \ v_i \ sinon$$

$$H_a = \{v | v \perp_{(v,a)=0} a = 0\}$$

$$v \in V, \ (a \neq 0)$$

Reflection:

$$S_a(v) = v - 2\frac{(v,a)}{(a,a)} \cdot a$$

(1) S_a est orthogonale

$$< S_a v, S_a w >=< v, w >$$

(2) $det(S_a) = -1$

(3) si $\gamma : V \to V$ orthoganal
(a) $\gamma H_a = H_{\gamma(a)}$
(b) $S_{\gamma(a)} = \gamma \circ S_a \circ \gamma^{-1}$

G un sous-groupe **fini** de $GL(V)$ Qui est eugendre par des reflections.

$$g \in G \ s_{a1}, ..., s_{at} \in G$$

$$g = s_{a1} \circ \ s_{a2} \circ ... \circ s_{at}$$

$\rho := s_b \circ s_a$ est une totation d'un angle egal A' 2θ ou' $cos(\theta) = (a, b)$

since G is finite there exist a k such that $\rho^k = Id$, $2k\theta = 2\pi$, $\theta = \frac{\pi}{k}$.

3.1 Exemples

(1) Dihedraux
(2) Group symetrique $(x_1, ..., x_n) \to (x_{\sigma(1)}, ..., x_{\sigma(m)})$
Engendre paz les transpositions $(i, j)(x_1, ..., x_i, ..., x_j, ..., x_m) = (x_1, .., x_{j1}, ..., x_i, ..., x_n)$ Est la reflexion Dans

$$H_{ij} = \{(x_1, ..., x_n) | x_i = x_j\}$$

$s_{ij} := e_i - e_j$ $e_i = (0, ..., 1, ..., 0)$

Exercise: *verifier que c'est le bun groupe* $\cong S_n$

(3) Groupe Hyperoctahedral
On cousidere la "'famille de vectors"'

$$\alpha = \{e_i - e_j, e_i + e_j, -e_i, e_i | 1 \leq i, j \leq n \ i \neq j\}$$

H_α Engendre un groupe, $2^n n!$ elements $\rightsquigarrow$ permutations signees.

3.2 Fair generaux sur les groupes "'de reflexion"'

G sous-grooupe de $GL(m,R)$ matrices $m \times m$

(1) Polynomes G-invariants $\quad P(gx) = p(x) \forall g \in G$
R^G L'anneeau de polynome $G - inv$.

Theorem 2 *(Hilbert Neother ...)*

$$R^G \cong R[f_1, f_2, ..., f_n] ou\ fi\ soni$$

Des Polynomes $G - invariants$ Homogenes. $f \in R^G$ ssi on peut ecrive. f de unique sous la forme

$$f = \sum a_{i,i2,...,ik} f_{i1}^{b_i} f_{i2}^{b_{i2}} ... f_{ik}^{b_{ik}}$$

mais les f_i ne sont **pas unique** cependant

$$d_1 := deg(f_1)....d_n := deg(f_m)$$

$$d_1 \leq d_2 \leq ... \leq d_n$$

est unique $\#G = d_1 d_2 \cdots d_n$ # De reflections $= (d_1 - 1) + (d_2 - 1) + ... + (d_n - 1)$ Dans G.
pour S_n
$d_1 = 1, d_2 = 2, ..., d_n = n$

$$\#G = n!$$

$$\#reflections = 0 + 1 + 2 + ... + (n-1) = \binom{n}{2}$$

Choix possibles pour les fi: (1)

$$e_1(x_1, ..., x_n) = x_1 + ... + x_n$$

$$e_2(x_1, ..., x_n) = x_1 x_2 + x_1 x_3 + ... + x_{n-1} x_n$$

....
(2) $h_1 = e_1,\ h_2 =_i^2 + \sum_{i<j} x_i x_j$

(3)

$$k_i = x_1^i + x_2^i + ... + x_n^i\ 1 \leq i \leq n$$

$$k_1, k_2, ..., k_n$$

sont algebraic independent. get from ameerah

Pour B_n $d_1 = 2, d_2 = 4, d_3 = 6, ..., d_n = 2n$

$$\#G = 2^n n!$$

$$\#reflections = 1 + 3 + 5 + ... + (2n-1) = n^2$$

$$\binom{n}{2} + \binom{n}{2} + n = n^2$$

$$R^{B_n} = \{f(x_1^2, ..., x_n^2) | f \in R^{S_n}\}$$

$$x_i \to -x_i$$

Chevaley-Sheppard-Todd-Bourbaki(50-60) 1 *G est un sous-groupe finie de $GL(R^n)$ engendre par reflexious (aussi pour $GL(C^n)$ aves adaptations) Geometrique* ***Engendre par des reflectious*** *ssi*
"'algebrique"' $R^G \cong R[f_1, ..., f_n]$ pur certains fi homog

equivalent

$$H_{R^G}(q) = \prod_{i=1}^{n} \frac{1}{(1-q^{d_i})} = \sum_{d=0}^{n} dim(R_d^G) q^d$$

$$R =_{d=0}^{\infty} R_d$$

Gradue action de group respecte le degne

$$x_i \to a_1 x_1 + a_2 x_2 + ... + a_n x_n$$

$$R^C = \bigoplus_{d=0}^{\infty} R_d^G$$

Pas un groupe de reflexion

$$H_{R^G}(q) = \frac{\sum s_j q^j}{\prod_{i=1}^{n} (1-q_i^{d_i})} \;\; a_j \in N$$

ssi R est libre comme $R^G - module$ ssi

$$dim(\frac{R}{I_G}) = \#G$$

I_G ideal eugendre par $f_1, f_2, ..., f_n$.

$$H_{\frac{R}{I_G}}(q) = \prod_{i=1}^{n} \frac{1-q^{di}}{1-q}$$

$$R \cong R^G \otimes \frac{R}{I_G}$$

ssi

$$\#G = d_1 d_2 ... d_n$$

ssi

$$dim(I_G^{\perp}) = \#G$$

$$I_G^{\perp} = R[\delta x^{\alpha} \Delta_G(x) | \alpha \in N^n]$$

ou

$$\delta_G(x) = \prod_{S_\alpha \ une \ reflection \ dans \ G} (x, \alpha)$$

produit scalaire

$$(x_1 a_1 + x_2 a_2 + ... + x_n a_n)$$

$$deg(\Delta_G(x)) = (d_1 - 1) + (d_2 - 1) + ... + (d_n - 1)$$

reference: Richard kane "'reflection groups and invariant theory"'
Richard Humphreys reflection groups and Coxter groups.

4 Dimension and Hilbert Series

Pour certaines generalisations des espaces coinvariants

Le Groupe Du jour: S_n

$$Q[x, y] \ x = x_1, x_2, ..., x_n$$

$$y = y_1, y_2, ..., y_n$$

$$S_n \times Q[x, y] \to Q[x, y]$$

$$\sigma x_i = x_{\sigma(i)}, \sigma y_i = y_{\sigma(i)}$$

$$f(x, y) \in Q[x, y]$$

$$\sigma \cdot f(x, y) = signe(\sigma) \cdot f(x, y)$$

$$L_\sigma[f(x, y)] := \{g(dx, dy) f(x, y) | g(x, y) \in Q[x, y]\}$$

Eugendre par(spanned by)

$$dx^a dy^b f(x, y)$$

$$dx_1^{a_1} dx_2^{a_2} ... dx_n^{a_n} dy_1^{b_1} ... dy_n^{a_n} f(x_1, ..., x_n; y_1, ..., y_n)$$

si on a $f(x)$, alons $L_d[f(x)] \subseteq Q[x]$

$$f(x, y) \in Q[x, y]$$

$$\sigma \cdot f(x, y) = signe(\sigma) \cdot f(x, y)$$

(1) $L[f(x,y)]$ est invariant our l'action de S_n si $g(x,y) \in L_d[f(x,y)]$ alons $\sigma g(x,y) \in L_d[f(x,y)]$

(2) $L_d[f(x,y)]$ est bihomogene et de dimension finie . Opinateur lineaure sur $Q[x,y]$

$$\prod_{kl}(x^a y^b) = x^a y^b \ si \ |a| = k \ and \ |b| = l \ 0 \ sinon$$

si $f(x,y)$ est bihomogene.
(3) Serie de Hilbert Bigraduee

$$H_n(q,t) := \sum_{k,l} dim(m_{k,l}) q^k t^l$$

$$\prod_{k,l} f(x,y) = \sum_{|a|=k,|b|=l} f_{ab} x^a y^b$$

Composante bihomogene de bidegne (k,l) de $f(x,y)$

Un sous-espace $V \subseteq Q[x,y]$ *bihomogene* ssi

$$\prod_{k,l}(V) \subseteq V$$

Pour tout $k,l \in N$
si $f(x,y) \in V$ alors $\prod_{k,l}(f(x,y)) \in V \ forallk,l$

$$f(x,y) = \sum_{k+l \leq deg(f)} \prod_{k,l}(f(x,y)) \ (dans \ Q[x,y])$$

Autrement dit

$$Q[x,y] = \bigoplus_{k,l} R_{k,l} \ ou \ R_{k,l} = \prod_{k,l}(R)$$

$Q[x,y]$ est bigradue par le bidegore. $R := Q[x,y]$

4.1 Example

(1) $\Delta_m(x) := det(x_i^{n-j})_{1 \leq i,j \leq n}$ M Harmoniques de S_n dimension $n!$, Serie de Hilbert $\prod_{k=1}^n \frac{1-q^k}{1-q}$

$$d_\lambda(q) \cdot \prod_{k=1}^n \frac{1-q^k}{1-q}$$

$$d_\lambda(q) \in N[q]$$

(B-Garcia-Tesler (2000))
(2)

$$det(x_i^{a_j})_{1 \leq i,j \leq n}$$

$$(a_1 > a_2 ... > a_n)_{:=\alpha} \geq 0, a_j \in N$$

$$:= \Delta_a(x)L[\Delta_a(x)]\ M_a$$

$$\lambda_j := a_j - (n-j) \in N$$

$$n-1 > n-2 > ... > 1 > 0$$

$$(\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n)_{Partage} \geq 0$$

Stanley 1979
let

$$\alpha = (5,4,3), \lambda = (3,3,3) = 3^3, n = 3$$

$$d_{k^n} = \sum_{\mu \subseteq k^n} q^{|\mu|}$$

$$\alpha = (k+n-1, ..., k+1, k)$$

$$\lambda = k^n := (k, k, ..., k)$$

$$d_{k^n}(1) = \binom{n+k}{n}$$

$$d_{k^n}(q) = \binom{n+k}{n}_q$$

<u>Exersize</u>

Montrer que

$$\sum_{\mu \subseteq k^n} q^{|\mu|} =$$

$$dim(M_{k^n + \delta n}) = (n+k)(n+k-1)\cdots(k+1)$$

<u>En General</u>

$$d_\lambda(q)$$

$$= dim(Q[S_{\lambda/\mu} | \mu \subseteq \lambda]) \leq \#\{\mu | \mu \subseteq \lambda\}$$

$$\mu_1 \leq \lambda_1, ..., \mu_k \leq \lambda_k$$

$S_{\lambda/\mu}$ fonction de Schur Gauches

$$S_{\lambda/\mu} := \sum_{T:Tableau\ semi-standart\ de\ forme\ \lambda/\mu} Z_t$$

$$z_i, 1 \leq i < \infty$$

$\delta - -partial\ dirivitive$

$$I_\alpha := \{f(x)|f(\delta x)\Delta_\alpha(x) = 0\}$$

$$= L_\delta[\Delta_\alpha(x)]^\perp$$

Remarque

$$\Delta_\alpha(x) = f_\alpha(x)\Delta_n(x)\ pour\ f_\alpha(x)\ polynome\ symmetrique$$
$$f_\alpha(x) = S_\sigma(x)$$

$$\sigma \cdot \frac{\Delta_\alpha(x)}{\Delta_n(x)} = \frac{\sigma\Delta_\alpha(x)}{\sigma\Delta_n(x)}$$

$$\frac{sign(\sigma)\Delta_\alpha(x)}{signe(\sigma)\Delta_n(x)} = \frac{\Delta_\alpha(x)}{\Delta_n(x)}$$

on cherche $g(x) \ni$

$$g(\delta x)\Delta_\alpha(x) = 0$$

si $g(x)$ est symmetric de degre assez graud

$$\cdot x = x \cdot \delta x + ID$$

operateurs

$$x \cdot (-) : f(x) \to xf(x)$$

(3) μ un paltage, $\mu \in N^2$ Ferrers diagram. $(i, j) \in \mu$ abus de langage (A)

$$S : (k, l) \ni k \leq i\ l \leq j$$

$$alurs(i, j) \in \mu \Rightarrow (k, l) \in \mu$$

(B)

$$\mu \subseteq N^2 \# \mu < \infty$$

Exemples

$$\Delta_\mu(x, y) := det(x_i^a y_i^b)_{1<i\leq b}$$

$(a, b) \in \mu$

$$\Delta_\mu(x, y) = det(1\ x_i\ y_[i\ x_i y_i)_{row\ i}$$

Theorem 3 $dim(L_\delta[\Delta_\mu(x, y)]) = n!$ (pas de preuve elementaire ou combinatoire.) Geometrique algebrique(¿ 101).

5 Theorie De La Representation(De S_n)

$$V = \bigoplus_{k,l \geq 0} V_{k,l}$$

Espace Vectoriel $dim(V_{k,l}) < \infty$.

$$S_n \times V \to V$$

Action Lineaire . Bigraduee'.
(1) $V \to^{\sigma}_{Linear} V$
(2) $\sigma(V_{k,l}) \subseteq V_{k,l} \rightsquigarrow S_n \times V_{k,l} \to V_{k,l}$
(3) $\sigma(\tau(-)) = (\sigma\tau)(-)$
(4) $Id(-) = Id_V$

Fait 1 Toute representation se deconpose (de facon unique)eu representations irreductibles.

Fait 2 Les Representations irreducible de S_n sont classifiees par les partage de n. Autoreuent dit. Si W est irreductible alors W est isomorphe a exactement un V_λ.

$\lambda \perp n$ (λ partage de n) ou' les V_λ sont cectains representants "'canoniques'" des rep irred.

Fait 3 on sait tout dune representation quand on commait son caractere

$$S_n \times W \to W \ ActionLinear$$

$$\chi_W : S_n \to C$$

$$\chi_w(\sigma) := Trace(\sigma(-))$$

Fait 4
$v \cong W$ ssi $\chi_V = \chi_W$
$\chi_{V \oplus W} = \chi_V + V_w$

(1) $\Leftrightarrow \chi_W = \sum_{\lambda \perp n} n_\lambda^w \chi_\lambda$
ou $\chi_\lambda := \chi_{V_\lambda}$

$n_\lambda^W \in N$
$n = P_1^{n1} P_2^{n2} ... P_k^{nk}, P_i \ Premiers$

5.1 Transforme de Frobenius

$$X : S_n \to C \rightsquigarrow fonctionsymmetric$$

$$X_1 + X_2 \rightsquigarrow somme$$

$$Produit\ scalar \rightsquigarrow scalar\ product$$

$$X_\lambda\ irreducible \rightsquigarrow Fonction\ de\ schur\ S_\lambda$$

Reformule

$$F_w = \sum_{\lambda \perp n} n_\lambda^W S_\lambda$$

Definition Directe du frobenius de W.

$$F_W := \frac{1}{n!} \sum_{\sigma \in S_n} Trace(\sigma(-)) h_{\lambda(\sigma)}$$

ou $\lambda(\sigma)$ = le partage qui decrit la structure cyclique de σ.

$$h_k = z_1^k + z_2^k + ...$$

Fonction Symetrique Somme de puissance(infinite de variables) Remarque
Les h_k sont algebriquement independants
Les h_λ sont linearement independants

σ est conjuge a τ

$$\sigma = \phi^{-1} \tau \phi \ \phi \in S_n$$

ssi

$$\lambda(\sigma) = \lambda(\tau)$$

$$Trace(\sigma(-)) = Trace(\tau(-))$$

Parce Que

$$Trace(P^{-1} AP) = Trace(A)$$

Il y une facon naturelle dindeter les representatious irreductibles de S_n de facon A ce Que

$$S_\lambda = \sum_{\lambda \perp n} \chi_\lambda(\sigma_\mu) \frac{h_n}{Z_\mu}$$

Ou σ_μ est ni importe quelle permutation telle que

$$\lambda(\sigma_\mu) = \mu\ "`structurecyclique"'$$

Exemple: $\mu = 3221$ On deut choisir

$$\sigma_n = (123)(45)(67)(8)$$

$$Z_\mu = 1^1 \cdot 1! s^2 2! 3^1 \cdot 1!$$

$$Z_\mu = 3^5 \cdot 5! 4^2 \cdot 2! 5^3 \cdot 3!$$

Supposons connus les $X_\lambda(\mu) := X_\lambda(\sigma_\mu)$ C'est a dire la matrice carre .
Voir Daus un bon livre ou avec maple.

(1) $\{S_\lambda\}_{\lambda \perp n}$ formeut une base de fonct sym. homogenius de degre n.

(2) Orthonormale pour le produit scalair

$$< h_\lambda, h_\mu >:= z_\lambda \ si \ \lambda = \mu \ \ 0 \ sinon$$

$$< S_\lambda, S_\mu >= 1 \ si \ \lambda = \mu, \ 0 \ sinon$$

$$n_\lambda^W =< F_W, S_\lambda >$$

Frobenius Bigradue(Bigraded)

$$F_W(q,t) := \sum_{k,l \geq 0} q^k t^l F_{V_{k,l}}$$

Donc

$$F_V(q,t) = \sum_{\lambda \perp n} n_\lambda(q,t) S_\lambda$$

ou $n_\lambda(q,t) = N[q,t]$

example:
$n_\lambda(q,t) = \frac{1}{1-(q+t)} = 1 + (q+t) + (q+t)^2 ...$
Combinatoirue.

Exemples:

$$F_{C[x,y]} = \sum_{\lambda \perp n} n_\lambda(q,t) S_\lambda$$

,

$C[x,y]$polynomes eu les variables

$$x_1, ..., x_n$$

$$y_1, ..., y_m$$

$n_\lambda(q,t) =? \ n_\mu(q,t)$ =coefficient de $S_{11...1}$ dans $F_{Q[x,y]}(q,t).S_{11...1}$ est le frobenius de la representation triviale $\sigma P(x,y) = P(x,y)$

$$Q[x,y] = \bigoplus_{\lambda \perp n} \bigoplus_{k,l \geq 0} u_{k,l}^\lambda$$

ou $U_{k,l}^\lambda$ est une srme directe de copies de V_λ.

$$n_m(q,t) = \sum_{k,l} n_{k,l}^{1^m} q^k t^l$$

ou

$$n_{k,l}^{1^n} = dim(\{P(x,y)|(x,y) = P(x,y) \ tout \ \sigma \in S_n \ et \ bihomogene \ de \ bideore(k,l)$$

Par exemple: $n = 3$ Bidegre : Dimension

$$(0,0):1,(1,0):1,(0,1):1,(1,1):2$$

we will write n for $11111..._{n\ ones}$
$m_n(q,t) =$ Coefficient de S_n dans $F_{Q[x,y]}(q,t)$

$$= \sum_{k,l\geq 0} m^n_{k,l} q^k t^l$$

$m^n_{k,l} = dim(\{P(x,y)|(x,y) = signe(\sigma)P(x,y),\ pur\ tout\ \sigma P(x,y)\ Bihomogene\ de\ bideone\ (k,l)$

$L_d[\Delta_\mu] =$ combinations lineaires des derivevs part de $\Delta_\mu(x,y)$ $\Delta_\mu(x,y) = det\{1\ x_i,\ y_i\}$ $i \in \{1,2,3\}$

$$= (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)$$

$$F_{M_\mu} = S_3 + (q+t)S_{21} + qtS_{111}$$

Dimension $= 6$ Un souse-espace de dimension 1 en bidegre $(1,1)$ constitue de polynome alternes (antisymetriques diaganaux)

$$= \{dtl\Delta_\mu(x,y)\}$$

$$\Delta_\mu$$

$$dx_1 \to y_2 - y_3$$

$$+dx_2 \to y_3 - y_1$$

$$+dx_3 \to y_1 - y_2$$

Theorem 4 *(2001 Haiman)*

$$F_{M_\mu}(q,t) = H_\mu(Z;q_1t)$$

(polynome symetriques de macdonald 1987. Auire definition via la theorie des fonctions sym.

6 Un calcul Elementaire De

$$F_{C[x]}(q) = \sum_{d\geq 0} q^d \frac{1}{n!} \sum_{\sigma\in S_n} Trace(\sigma\cdot(-)|_{R_d})p_\lambda(\sigma)$$

$$R := C[x], R =_{d\geq 0} R_d$$

Remarque 1:

$$\frac{1}{n!}\sum_{\sigma\in S_n} f_\sigma p_{A(\sigma)} = \sum_{\mu\perp n} f_\mu \frac{p_\mu}{z_\mu}$$

Avec $f_\sigma = f_\tau$ si $\sigma = \phi^{-1}\tau\phi$ pour un $\phi \in S_n$ $(\lambda(\sigma) = \lambda(\sigma))$. $\frac{n!}{Z_\mu}$est le nombre de τ conjogue A σ, si $\lambda(\sigma) = \mu$. $\mu = 1^{d_1}2^{d_2}...n^{d_n}$ $\sum_k d_k = n$ $d_i =\#$ cycles de long $i$dans σ

$$Z_\mu = 1^{d_1}d_1!2^{d_2}d_2! \cdots n^{d_n}d_n!$$

$$= \sum_{\mu \perp n} \frac{p_\mu}{Z_\mu}(\sum_{d \geq 0} Trace(\sigma(-)|_{R_d})q^d)$$

Base de R_d contitutee des X
$tesque—\alpha| = d$

$$^\alpha = x^{a_1}_{\sigma(1)}x^{a_2}_{\sigma(2)} \cdots x^{a_n}_{\sigma(n)} = X^\beta$$

$$Trace(\sigma(-)|_{R_d}) = \#\{x^\alpha|\sigma^\alpha = x^\alpha\}$$

$$p_\mu = p_1^{d_1}p_1^{d_1} \cdots p_n^{d_n}$$

$$= \sum_{\mu \perp n} \prod_{i=1}^n (\frac{p_i}{i(1-q^i)})^{di} \cdot \frac{1}{d_i!}$$

6.1 Substitution Plethystique

S_n Fouction de schur associee au partage

$$S_n = \sum_{\mu \perp n} \prod_{i=1}^n (\frac{p_i}{i})^{d_i})\frac{1}{d_i!}$$

"‘Operation "‘ sure les fonction symetrique, "‘evaluation"’ des.
Definient comme suit: Regles de calcul

$$p_k[x+y] := p_k[x] + p_k[y]$$

$$p_k[x \cdot y] := p_k[x] \cdot p_k[y]$$

$$p_k[q \cdot x] = q^k p_k[x]$$

$$p_k[ax] = ap_k[x]$$

q variable, a cte : Attention les calculus ne commutent pas avec l'evaluation.
si f et g sont fonction symetriques alors

$$(fg)[x] = f[x]g[x]$$

$$(af + bg)[x] = af[x] + bg[x]$$

Penser Que p_k est l'operation qui consiste A' elever A' la puissause k les variables.

$$p_k[x_1 + x_2 + ... + x_n] = x_1^k + x_2^k + ... + x_n^k$$

$$Z = z_1 + z_2 + ...$$

$$p_k[\frac{z}{1-q}] = p_k[Z \cdot (1 + q + q^2 + ...)]$$

$$= p_k[\sum_{i,j} z_i q^j]$$

$$= \sum_{i,j} z_i^k q^{kj}$$

$$= \frac{p_k[Z]}{1-q^k}$$

6.2 Calcul des fonctions symetriques(∃ preuve combinatoire)

$$S_n = \sum_{\mu \perp n} \frac{p_\mu}{Z_\mu}$$

$$S_\lambda = det(S_{\lambda_i + j - i}) i\ Ligne, j\ colones$$

$$S_{332} = det(S_{\lambda_i - j - i}\ i\ Ligne\ j\ colones$$

$$\begin{pmatrix} S_3 & S_4 & S_5 \\ S_2 & S_3 & S_4 \\ S_0 & S_1 & S_2 \end{pmatrix}$$

$$S_{332} = S_3^2 S_2 + S_4^2 + S_5 S_2 S_1 - S_4 S_3 S_1 - S_4 S_2^2 - S_5 S_3$$

Theorem 5

$$< S_\lambda, S_\mu >= (1\ si\ \lambda - M\ 0\ ou < p_\lambda, p_\mu >:= Z_\mu$$

Schure Gauches definient par $< S_{\lambda/\mu}, S_V >=< S_\lambda, S_\mu S_V >$
Surprise
Non-Surprise $S_\mu S_V = \sum_{\lambda \perp |\mu| + |V|} c_{\mu V}^\lambda S_\lambda$
Surprise $c_{\mu v} \in N$!! Coeff de littlewood-Richardson.

$$< S_\mu S_V, S_\lambda >= c_{\mu v}^\lambda$$

$$S_{\lambda/\mu} = \sum_{v \perp |\lambda| - |\mu|} c_{\mu v}^\lambda S_v$$

6.3 Formules

$$S_\lambda[x+y] = \sum_{\mu \subseteq \lambda} S_\mu[x] S_{\lambda/\mu}[y]$$

$$S_\lambda[x] = (x^n \; si \lambda = (n) \; 0 \; sinon$$

$$S_\lambda[x^n] = det(A)$$

$$A := \begin{pmatrix} S_{\lambda_1}[x] & S_{\lambda_1+1}[x] & ... \\ S_{\lambda_2-1}[x] & S_{\lambda_2}[x] & ... \\ .. & ... & ... \\ .. & \searrow & ... \\ ... & ... & S_{\lambda_k}[x] \end{pmatrix}$$

$$\lambda_1 \geq \lambda_2 ...$$

$$\S_\lambda[x_1 + x_2 + ... + x_k] = 0 \; si \; k < \#parts \; de \; \lambda$$

$$S_{\frac{\lambda}{\mu}}[x] = 0 \; si \; \frac{\lambda}{\mu} \; contient$$

$$S_\lambda[(x_1 + x_2 + ...) + x_k] = \sum_{T \; Tableau \; semi-standard \; de \; forme \; \lambda} X_T$$

6.4 Bases Duales

$$\Lambda_d := \; combinaisons \; lineaires \; de \; p_n, \mu \perp d$$

$dim(\Lambda_d) = \#$ partages de d.
$\{S_\lambda\}_{\lambda \perp d}$ base orthonormale

$$< p_{\lambda_1}, p_\mu >:= \delta_{\lambda\mu} Z_\mu$$

$$\{U_\lambda\}_{\lambda \perp n} \; et \; \{V_\mu\}_{\mu \perp n} \; sont \; des \; bases \; duales \; ssi$$

$$< U_\lambda, V_\mu >= \delta_{\lambda\mu} \; \; (Delta \; de \; kronecker)$$

Prop: (Exersize diag. Lin)

$$\{U_[\lambda\}_{\lambda \perp n} \; et \{V_\mu\}_{\mu \perp n} \; sont \; Duales$$

ssi(*) (*) iff

$$S_n[xy] = \sum_{\lambda \perp n} U_\lambda[x] V_\lambda[y]$$

(*)

$$S_n[xy] = \sum_{[} \lambda \perp n U_\lambda[x] V_\lambda[y]$$

Avec $U_\lambda[x] = p_\lambda[x]$

$$V_\mu[x] = \frac{p_\mu[x]}{Z_\mu}$$

$$puisque\ S_n = \sum_{\lambda \perp n} \frac{p_\lambda}{Z_\lambda}$$

$$p_\lambda[xy] = p_{\lambda][x] p_\lambda[y] < p_\lambda, \frac{p_n}{Z_\mu} > = \delta_{\lambda\mu}}$$

1 consequence

$$S_n[\frac{Z}{1-q}] = \sum_{\lambda \perp n} S_\lambda[\frac{1}{1-q}] S_\lambda[Z]$$

Cas Special $\lambda = (n)$

$$S_n[\frac{1}{1-q}] = ?$$

$$S_3 = \frac{p_1^3}{6} + \frac{p_2 p_1}{2} + \frac{p_3}{3}$$

$$S_3[\frac{1}{1-q}]$$

$$= \frac{1}{6}(\frac{1}{1-q})^3 + \frac{1}{2}(\frac{1}{1-q^2})(\frac{1}{1-q}) + \frac{1}{3}\frac{1}{(1-q^3)}$$

$$= \frac{1}{(1-q)(1-q^2)(1-q^3)}(\frac{(1+q)(1+q+q^2)}{6} + \frac{(1-q^3)}{2} + \frac{(1-q)(1-q^2)}{3}$$

$$= \frac{1}{(1-q)(1-q^2)(1-q^3)} \cdot\ 1$$

$$= \frac{1}{(1-q)(1-q^2)(1-q^3)} S_3 + ...$$

$R = C[x],\ R \cong R^{S_n} \otimes R_{S_n}$ isomorphism de representation

$$S_n[\frac{Z}{1-q}] = F_R(q) = (\prod_{k=1}^{n} \frac{1}{1-q^k}) \cdot F_{R_{S_n}}(q)$$

7 new thing

$$S_n[x] := \sum_{|\alpha|=k} x^\alpha$$

$$x = x_1 + x_2 + ... + x_n \ aussi \ n \to \infty$$

$$\sum_{k\geq 0} S_k[x]z^k = \prod_{i=1}^{n} \frac{1}{1 - x_i z}$$

$$= exp(log(\prod_{i=1}^{n} \frac{1}{1 - x_i z})) = exp(\sum_{i=1}^{n} log(\frac{1}{1 - x_i z}))$$

$$= exp(\sum_{i=1}^{n} \sum_{j=1}^{\infty} \frac{(x_i z)^j}{j})$$

$$= exp(\sum_{j=1}^{\infty} \frac{z^j}{j} (\sum_{i=1}^{n} x_i^j))$$

$$= exp(\sum_{j=1}^{\infty} \frac{z^j}{j} p_j[x])$$

$$S_n[xy] = \sum_{\lambda \perp n} u_\lambda[x] v_\lambda[y]$$

ssi

$$< u_\lambda, v_\mu >= \delta_{\lambda\mu}$$

$$xy = (x_1 + x_2 + ...)(y_1 + y_2 + ...) = \sum_{i,k} x_i y_k$$

$$\sum_{k\geq 0} S_k[xy]z^k = \prod_{i,k} \frac{1}{1 - x_i y_k z} (Noyau \ De \ Cauchy)$$

$$= exp(\sum_{j=1}^{\infty} \frac{z^j}{j} p_j[xy]$$

$$\Omega := exp(\sum_{j=1}^{\infty} \frac{p_j}{j}$$

'un operateur'

$$p_j[zxy] = z^j p_j[xy]$$

p_j est l'operateur $\ni$

$$p_j[x_1 + x_2 + ...] = x_1^j + x_2^j + ... + x_n^j$$

7.1 Involution ω

ω lineaire et multiplicatif, $\omega(p_k) := (-1)^{k-1}p_k$.
Properties
(1) $\omega^2 = ID$
(2) $\omega(S_\lambda) = S_{\lambda'}$($\lambda'$ is the transpose of the diagram of λ).
Cas special:$\omega(S_n) = S_{111\ldots1_{n-fois}}$

Exersice:Prover (2) en utilisant Jacobi-Trudi.

$$\Omega[x+y] = \Omega[x] \cdot \Omega[y]$$

$$\Omega[-x] = \frac{1}{\Omega[x]}$$

8 Polynomes De MacDonald

DePart $n = 4$

$m_{1111}, m_{211}, m_{22}, m_{31}, m_4$.
$m_\alpha[x] = Some\ de\ monome\ x^v\ tels\ que\ \alpha\ est\ la\ liste\ obtenue\ en\ triant\ v\ de\ facon\ decroissante.$

$$x^v = x_1^{v_1} x_2^{v_2} \cdots$$

$$v = (0, 4, 2, 1, 5, 0, 0, 3, 2, ...)$$

$$\alpha = 54321000...$$

Ordre croissant lex. Des Partages de 'n'.

$$< p_\lambda, p_\mu >_{q,t} := (\prod_{i=1}^{l} \frac{(1 - q^{\lambda_i})}{1 - t^{\lambda_i}}) z_\lambda\ si \lambda = \mu\ 0\ si\ \lambda \neq \mu$$

Algorithm D'orthogonalisation De Gram-Schmidt. Resultat: $P_\mu[z; q, t]$
Specialisations:
$q = t = 0$ $P_\mu[z; 0, 0] = S_\mu$
$q = 0$ $P_\mu[z; 0, t]$ $Hall - Littlewood$

$q = t^\alpha + lim_{t\to1}$ Jack
Clairement unde base

$$P_{31} = m_{31} + \frac{a_{22}(q,t)}{b_{22}(q,t)} m_{22} + \frac{a_{211}(q,t)}{b_{211}(q,t)} m_{211} + \frac{a_{1111}(q,t)}{b_{1111}(q,t)} m_{1111}$$

$a_\lambda, b_\lambda \in Q[q, t]$

$\lambda \geq \mu$ ssi $\lambda_1 \geq \mu_1$, $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$ $\lambda_1 + \lambda_2 + \lambda_3 \geq \mu_1 + \mu_2 + \mu_3$...

Theorem 1 *Il existe(explicite) un operateur lineaire sur l'espace des foncti sym pour lequel les polynomes de macdonald sont des fonctions profres avec des valeurs profres distinctes.*

Proposition 5 *Le result dans la definition est le meme quelque soit l'extension lineaire de l'ordre de la dominance choisit pour executer L'algorithm.*

$$P_\mu = m_\mu + \sum_{\lambda \leq \mu} \frac{a_\lambda(q,t)}{b_\lambda(q,t)} m_\lambda$$

$$P_\mu = \sum_{\lambda \perp n} \frac{a'_\lambda(q,t)}{b'_\lambda(q,t)} p_\lambda$$

8.1 See Handout Chapter 9

$$H_\mu[z;1,1] = \sum_{\lambda \perp n} f_\lambda S_\lambda =_{RSK correspondence} (S_1)^n = (z_1 + z_2 + ...)^n$$

8.2 Approche Combinatoire(Pure)

$$H_\mu[z;q,t] = \sum_{f:\mu \to N^+ q^{inv(f)} t^{maj(f)} z_f}$$

z_f : fonc symetric monomial. Consequence

$$= \sum_{\lambda \perp n} G_{\lambda\mu}(q,t) m_\lambda$$

$$G_{\lambda\mu](q,t) \in N[q,t]}$$

$$maj(f) = \sum_{c \in Des(f)} l(c) + 1$$

$$Inv(f) = \{(c,d) | f(c) > f(d) \ d \ dans \ la \ region\}$$

$$inv(f) = \#Inv(f) - \sum_{c \in Des(f)} a(c)$$

8.3 Representations Bigraduees

$$F_{L_d[\Delta_\mu]}(q,t) = H_\mu[z;q,t]$$

$$\sum_{\lambda \perp n} k_{\lambda\mu}(1,1) S_\lambda = (S_1)^n$$

$$\Rightarrow dim(L_d[\Delta_\mu]) = n!$$

$$= \sum_{\lambda \perp n} k_{\lambda\mu}(q,t) S_\lambda$$

$\uparrow$ Multiplicite Bigraduees de rep irreducibles.

$$M_\mu := L_\delta[\Delta_\mu]$$

$$\Delta_\mu = det(x_i y_i^b)_{(a,b)\in\mu \ 1\leq i\leq n}$$

$$M_n = L_\delta[\Delta_n] = I_n^\perp$$

$$I_n = (f(x)|f \ sym \ and \ f(0) = 0)$$

$$dim(M_n) = n!$$

$$F_{m_n}(q,t) = \prod_{k=1}^{n}(1-q^k) S_n[\frac{z}{1-q}]$$

$$Q[x] \cong Q[x] \otimes M_n$$

Isomorphisme De $S_n - module$

M_n Polynomes Harmoniques pour $S_n \cong \frac{Q[x]}{I_n}$.

$$\Delta_n(x,y) = \prod_{i>j}(x_i - x_j)$$

$$S_n = \sum_{\lambda \perp n} \frac{p_\lambda}{z_\lambda}$$

$$S_n[\frac{z}{1-q}] = \sum_{\lambda \perp n} \frac{1}{z_\lambda} \prod_{k=1}^{n}(1-q^k) p_\lambda[\frac{z}{1-q}]$$

$$z = z_1 + z_2 + ...$$

$$\sum_{\lambda \perp n} k_{\lambda_1}(n)(q,t) s_\lambda = \prod_{k=1}^{n}(1-q^k) S_n[\frac{z}{1-q}]$$

$$= \sum_{\lambda \perp n} \frac{1}{z_\lambda} \prod_{k=1}^{n} (1-q^k) p_\lambda[\frac{z}{1-q}]$$

$$\downarrow lim_{q\to 1} \frac{1}{z_\lambda}(1-q)^n \prod_{k=1}^{n}(1+...+q^{k-1}) \prod_{i=1}^{l(\lambda)} \frac{p_{\lambda_i}}{(1-q)(1+q+...+q^{\lambda-1})}$$

$$= p_1^n$$

$$S_n[\frac{z}{1-q}] = \sum_{\lambda \perp n} S_\lambda[\frac{1}{1-q}] S_\lambda$$

$$\prod_{k=1}^{n}(1-q^k) S_\lambda[\frac{1}{1-q}] = \sum_{T\ tableau\ standard\ de\ forme\ \lambda} q^{c(T)}$$

$$F_\mu(q,t) = \sum_{\lambda \perp \mu} k_{\lambda\mu}(q,t) S_\lambda$$

$$k_{\lambda\mu}(q,t) \in N[q,t]$$

$$=_{thm} H_\mu[z;q,t]$$

$$\mu \rightsquigarrow \mu' \ \Delta_\mu(x,y) = \Delta_{\mu'}(y,x)$$

$B(f) = (deg_x(f), deg_y(f))$ $k_{\lambda\mu}(q,t) \in N[q,t]$
$f(x,y)$ *Bihomogene*
$\delta x^\alpha \delta y^\beta \Delta_\mu$ *est bihomogene*

$$(1)\ F_\mu(q,t) = F_{\mu'}(t,q)$$

$$(2)\ < S_n, F_\mu >= 1$$

$$(3)\ \downarrow_\mu f(x,y) := f(\delta x, \delta y)\Delta_\mu(x,y)$$

$$(1)\ \downarrow_\mu : M_\mu \to M_\mu\ inversible$$

$$(2)\ si\ f(x,y)\ est\ Bihomogene\ de\ bioegene\ (k,l)$$

$$alors \downarrow_\mu f(x,y)\ est\ de\ bidegre\ (n(\mu')-k, n(\mu)-l)$$

$$(3) \downarrow_\mu\ est\ un\ anti-isomorphisme\ de\ representation$$

$$< S_\mu, F_\mu(q,t) >= q^{n(\mu')} t^{n(\mu)}$$

$$F_{\downarrow M_\mu}(q,t) = q^{n(\mu')}t^{n(\mu)}\omega F_{M_\mu}(q^{-1},t^{-1})$$
$$\downarrow_\mu f(x,y)\Delta_\mu(x,y) = 0$$
$$\Rightarrow < f, \Delta_\mu >= 0$$
$$f \in M_\mu^\perp$$

$$(4)\ \ < S_n, F_\mu >= q^{n(\mu')}t^{n(\mu)}$$

$$(5)\ F_\mu(0,1) = h_\mu := S_{\mu 1}S_{\mu 2}...S$$

Livre: Bruce Sagan.

9 Poynomes Harmoniques Diagonalix

$x = x_1, ..., x_n$, $y = y_1, ..., y_n$
$S_n \times R \to R$, $R = Q[x,y]$
$\sigma \cdot x_i = x_{\sigma(i)}$
$\sigma \cdot y_i = y_{\sigma(i)}$

Une Base Algebrique:

$$p_{k,j}(x,y) := \sum_{i=1}^{n} x_i^k y_i^j$$

Est-ce que $\{p_{k,j}(x,y)\}_{k+j\leq n}$ est une base algebrique , etalgebriquement independants? L'ideal engendre par les $p_{k,j}(x,y), k+ \neq 0$. Est denote J_n
$D_n := J_n^\perp < f(x,y), g(x,y) >:= f(\delta x, \delta y)g(x,y)_{x=0,y=0}$

$$D_n = \{f(x,y)|\sum_{i=1}^{n} {}_i^k \delta y_i^l f(x,y) = 0, k+l > 0\}$$

Proposition 6 *Pour Tout Partage μ de n , on a $\Delta_\mu(x,y) \in D_n$. Et Donc $M_\mu \subseteq D_n$.*

Faits:
(1) $dim(D_n) = (n+1)^{n-1}$
(2) Si on Pose $AD_n := \{f(x,y) \in D_n|\sigma f(x,y) = signe(\sigma)f(x,y)\}$
($\Delta_\mu \in AD_n$) Alors $dim(AD_n) = \frac{1}{n+1}\binom{2n}{n}$.
(3) $q^{\binom{n}{2}}H_{AD_n}(1, \frac{1}{q}) = \frac{1}{[n+1]}\binom{2n}{n}_{q-analog}$

$$H_{d_n}(q,t) := \sum_{k,l} q^k t^l dim(\pi_{k,l}D_n)$$

(Stanley)

$$H_{AD_3}(q,t) = q^3 + q^2 + qt^2 + t^3 + qt$$

$$= q^3(q^3 + \frac{q^2}{q} + \frac{q}{q^2} + \frac{1}{q^3} + \frac{q}{q}$$

$$= q^6 + q^4 + q^2 + 1 + q^3 =$$

$$\frac{1}{(1+q+q^2+q^3)} \cdot \frac{(1+q+q^2+q^3)(1+q+q^2+q^3+q^4)(1+q+q^2+q^3+q^4+q^5)}{(1+q)(1+q+q^2)}$$

(4) $q^{\binom{n}{2}} H_{D_n}(q, \frac{1}{q}) = [n+1]^{n-1}$

Theorem(M.Haiman 2002):

$$F_{D_n}(q,t) = \nabla(S_{111\ldots1})$$

Def:

$$\nabla : \Lambda_{Q(q,t)] \to \Lambda_{Q(q,t)} \ lineaire \nabla H_\mu[z,q,t] := q^{n(\mu_1)} t^{n(\mu)} H_\mu[z;q,t]}$$

$n = 3, H_3 =$:

$$\begin{bmatrix} & 3 & 21 & 111 \\ 3 & 1 & q^2+q & q^3 \\ 21 & 1 & q+t & qt \\ 111 & 1t^2+t & t^3 & \end{bmatrix}$$

$$H_{21} = S_3 + (q+t)S_{21} + qtS_{111}$$

$$\nabla_3 = H^{-1}AH_3$$

$$A := \begin{bmatrix} q^3 & 0 & 0 \\ 0 & qt & 0 \\ 0 & 0 & t^3 \end{bmatrix}$$

now:

$$C_n(q,t) =< \nabla(S_{1^n}), S_{1^n} >= H_{AD_n}(q,t)$$

$q, t - Catalan$

Proposition 7 *L'espace AD_n est le plus petite space qui contient $\Delta_n(x)$ et qui ferme pour les operateurs*

$$E_k := \sum_{i=1}^{n} y_i{}_i^k$$

Observation: $E_k f(x,y)$ est diagonalement alternant si $f(x,y)$ l'est.

10 Operateurs

$p_{k,j}(\delta x, \delta y) = \sum_{i=1}^{n} \delta x_i^k \delta y_i^j$

$$E_l = \sum_{i=1}^{n} y_i \delta x_i^l$$

$$F_l = \sum_{i=1}^{n} x_i \delta y_i^l$$

$$[E_l, p_{k,j}(\delta x, \delta y)] = E_i p_{kj} - p_{k,j} \circ E_l =$$

$$[\delta y^l, y] = (\delta y^l \cdot y - y \cdot \delta y^l) f(y) = \delta y^l y \cdot f(y) - y \delta y^l f(y)$$

$l = 1$

$$\delta^2 y(yf(y)) = \delta y(f(y) + y \cdot \delta y f(y))$$
$$= 2\delta y f(y) + y \delta y^2 f(y)$$

$$[\delta y \cdot y, y \cdot \delta y] = ID$$

$$[\delta y^2, y] = 2\delta y$$

$$[p_{k,j}, E_l] = (\sum_i \delta x_i^k \delta y_i^j) \circ (\sum_i y_i \delta x_i^l) - (\sum_i y_i \delta x_i^l) \circ (\sum_i {}_i^k \delta y_i^j)$$

$$= \sum_i [\delta x_i^k \delta y_i^j, y_i \delta x_i^l]$$

$$= \sum_i [\delta y_i^j, y_i] \circ \delta x_i^{k+l} = \sum_i j \delta y_i^{j-1} \delta x_i^{k+l}$$

$$p_{k,j} E_a E_b = E_a p_{k,j} E_b + j p_{k+a,j-1} E_b$$

$$= E_{ab} p_{k,j} + j E_a p_{k+b,j-1} + j E_b p_{k+a,j-1} + j(j-1) p_{k+a+b,j-2}$$

10.1 Polynome Quasisymetriques

$$M_{132}(x,y,z,t) = xy^3z^2 + xz^3t^2 + yz^3t^2 + xy^3t^2$$

Theorem 2

$$dim(\frac{Q[x]}{(Qsym_n^+)}) = \frac{1}{n+1} 2n$$

11 topics

(1) Preuve de $< S_{\lambda_1}, S_{\mu/v} >=< S_\lambda S_v, S_\mu >$.
(2) Dualite' de shchor-Weyl $GL(n, C)$ S_n
(4) Paires de Gelfrand and Polynomes Zonaux Spheriques.
(5) Algebres de Hecke and Polynomes de Kazdan-Lusztig.

12 q-Inversion de Lagrange

$f(z) = z + f_2 z^2 + f_3 z^3 + ...$
$F(z) = z + g_2 z^2 + g_3 z^3 + ...$

$$f(F(z)) = z = f \circ_q F(z)$$

$$F(f(z)) = z = F \circ_{q^{-1}} f(z)$$

$$f(z) = \frac{z}{E(z)}$$

Lorsque

$$E(z) = 1 + e_1 z + e_2 z^2 + ...$$

$$z = f(F(z)) = \frac{F(z)}{E(F(z))}$$

$$****$$

$$F(z) = zE(F(z))$$

$$****$$

Une formule pour g_k en terme de e_j.

$$****$$

$$g_k = \sum_{c \ Chemin \ de \ Dyck \ De \ hauteur \ k} e_c$$

$$****$$

Solution Generique.
$e_c = e_1^2 e_2^2$

$$\nabla(e_k)|_{q=t=1} = g_{k+1}$$

$$(f \circ_q F)(z) = \sum f_k (F(z))^k$$

$$(f \circ_q F)(z) = \sum_{k \geq 1} f_k F(z) F(qz) ... F(q^{k-1} z)$$

12.1 Example:

$$C(z) = 1 + zC^2(z)$$

$$zC^2 - C + 1 = 0$$

$$C(z) = \frac{1 - \sqrt{1-4z}}{2}$$

$$C(z) = 1 + 2z + 5z^2 + 14z^3 + ...$$

$$= \sum \frac{1}{n+1}\binom{2n}{n} z^{n-1}$$

now

$$C(z) = 1 + zC(z)C(qz)$$

$$C(z) = \sum_{n\geq 0} C_n(q)z^n$$

$$C_n(q) = 1 \ si \ n = 0 \quad \sum_{k=0}^{n-1} q^k C_k(q)C_{n-1-k}(q)$$

end example

$$F(z) = z(E \circ_q F)(z)$$

Proposition 8 *La solution de $F = z(E \circ_q F)$ est $F(z) = \sum_{n\geq 0} q^n g_n(q) z^n$ ou*

$$g_n(q) = \sum_{\mu \perp n} q^{n(\mu_1)} h_\mu[\frac{z}{1-q}] f_\mu[1-q]$$

Fonction sym A' ecrique en terme des e_j

$$h_{3321} = h_3h_3h_2h_1$$

$$h_k = S_{(k)}$$

$$h_{21}[\frac{z}{1-q}] = h_2[\frac{z}{1-q}]h_1[\frac{z}{1-q}]$$

$$h_k = \sum_{\lambda \perp k}$$

$$p_k \to \frac{p_k}{1-q^k}$$

$$h_{21}[\frac{z}{1-q}] = \frac{1}{2}[(\frac{p_1}{(1-q)})^2 + \frac{p_2}{1-q^2}]\frac{p_1}{1-q}$$

$$p_1 = e_1$$
$$p_2 = e_1^2 - 2e_2$$

$$e_k = \sum_{\lambda \perp k} (-1)^{k-l(\lambda)} \frac{p_\lambda}{z_\lambda}$$

$$< p_\lambda,$$

$$< e_\lambda, f_\mu >= \delta_{\lambda\mu}$$

Posons

$$G(z) = exp(\sum_{k \geq 1} (-1)^{k-1} \frac{p_k z^k}{(1-q^{-k})k})$$

Alors

$$\star : E(z)E(z/q)E(z/q^2)...E(z/q^{n-1}) = \frac{G(z)}{G(z/q^n)}$$

$$E(z) = exp(\sum_{k \geq 1} (-1)^{k-1} p_k \frac{z^k}{k})$$

$$\star = \exp(\sum_{j=0}^{n-1} \sum_{k \geq 1} (-1)^{k-1} p_k (\frac{z}{q^j})^k / k)$$

$$= exp(\sum_{k \geq 1} (-1)^{k-1} p_k z^k / k \sum_{j=0}^{n-1} \frac{1}{q^{jk}})$$

$$\sum_{j=0}^{n-1} \frac{1}{q^{jk}} = \sum_{j=0}^{\infty} \frac{1}{q^{jk}} - \frac{1}{q^{nk}} \sum_{j=0}^{\infty} \frac{1}{q^{jk}}$$
$$= \frac{1}{1-q^{-k}} - \frac{1}{q^{nk}} \frac{1}{1-q^{-k}}$$

$$f \circ_q F \quad F \circ_{q^{-1}} f$$

On cherche les $q_k(q)$ tels que

$$\sum_{k \geq 1} \frac{f_k z^k q^{\binom{k}{2}}}{E(z)E(z/q)...E(z/q^{n-1})} = z$$

$$f = \frac{z}{E(z)}$$

$$F \circ_{q^{-1}} f = z$$

$$\sum_{k\geq 1} q_k(q) q^{-\binom{k}{2}} G(z/q^k) z^k = zG(z)$$

$$\Gamma(z) = \sum_{n\geq 0} \alpha_n(q) q^{\binom{n}{2}} z^n$$

ou

$$G(z) = \sum_{n\geq 0} \alpha_n(q) z^n$$

$$F(z)\Gamma(z) = z\Gamma(zq)$$