Solutions to Assignment1

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1 Exercise 7.14

(a) We are turning this problem of finding a feasible "evacuation plan" to a problem of finding a maximum flow in some network flow. Then one can apply Ford-Fulkerson algorithm to find the maximum flow.

Let G' be a network flow with the following setting. We assign capacity one to every edge in the initial graph G and we add a source s and a sink t to G. We also add edges $\overrightarrow{(s, x_i)}$ for every $x_i \in X$ of capacity one and we add edges $\overrightarrow{(s_i, t)}$ of capacity $k := |X|$ for every $s_i \in S$.

Claim 1.1. *There exists a feasible "evacuation" if and only if the maximum flow from s to t is equal to $k := |X|$*

Proof. $\Rightarrow$ If there exists a feasible "evacuation" then we can send $|X|$ units of flow from s and then send a unit flow from each x_i to some s_j through some path and from these s_j's c_j units of flow to t, where c_j is the the number of x_i's that evacuate to this specific "safe place" s_j. At the end we get a flow in G' of value k.

$\Leftarrow$ The converse is easier to see. Indeed, if the maximum flow is of value k, then there exists an integral one, therefore just by forgetting about s and t one can get k edge-disjoint paths from X to S.

(b) Note that the solution for part (a) will not work for this one, therefore we need to make sure that paths from X to S are vertex-disjoint. To do that, note that for every node x, we can replace it with two nodes x^+ and x^- and all edges that are of form $\overrightarrow{(u, x)}$ replace with $\overrightarrow{(u, x^-)}$ and $\overrightarrow{(x, u)}$ ones with $\overrightarrow{(x^+, u)}$(see Figure 1). We assign all these edges capacity one.

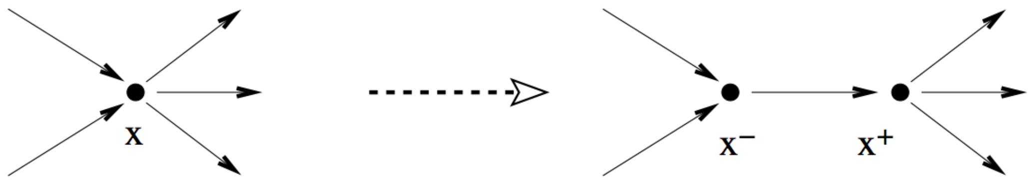

Figure 1: Finding vertex-disjoint paths

A claim similar to the Claim 1.1 is valid for this case too, one can find vertex-disjoint paths from X to S if and only if the maximum flow from X to S is equal to k.

Remark: Note that it is possible for a graph G to have edge-disjoint paths from X to S but not vertex-disjoint ones. Indeed, let $X = \{x_1, x_2\}$ and $S = \{s_1, s_2\}$ and $V(G) = S \cup X \cup \{v\}$. The adjacency of the nodes is as follows: $\overrightarrow{(x_i, v)}$ and $\overrightarrow{(v, s_i)}$ for $i = 1, 2$. It is easy to see that in this graph there are edge-disjoint paths from X to S , but not vertex-disjoint paths. $\square$

2 Exercise 7.28

(a) We are going to build a network flow and show that finding a proper TA hours scheduling is equivalent to the problem of finding a feasible circulation in this network. Let $G = (V, E)$ be the following network. Create a node u_i for each TA i and a node v_j for each time slot I_j, join u_i to v_j with a directed edge $\overrightarrow{(u_i, v_j)}$ and give capacity equal to 1 if and only if TA i is available for time slot I_j. Then add a source s in G that is connected with an edge $\overrightarrow{(s, v_i)}$ of lower bound a and capacity b to every v_i, and add a sink t which is connected with an edge $\overrightarrow{(u_j, t)}$ of capacity 1 to every u_j. Let the demand of s to be $-c$ and of c to be t, all other nodes have zero demands.

Claim 2.1. *There exists a proper TA scheduling if and only if there exists a feasible circulation in G.*

Proof. $\Rightarrow$ If there exists a proper TA scheduling, that means, we can send one unit of flow from each v_j to some u_i (since every TA works at least a hours) and in total the value of the flow must be c, since there are at least c hours being held every week. Also note that since since no two TA work during the same office hour, there will be no u_j receiving flow greater than one unit, therefore the flow will satisfy the capacities of edges from $\overrightarrow{(u_j, t)}$. This flow is feasible since the demand of t is equal to c.

$\Leftarrow$ The converse is also easy to see. A feasible circulation means that each v_j receives at least a units of flow, which corresponds to jth TA to hold at least a office hours and the capacity b of the edge $\overrightarrow{(s, v_i)}$ guarantees that no TA works more than b hours. Since the capacities of the edges $\overrightarrow{(u_j, t)}$ is one, no two TA will work at the same time (since no u_j receives two units of flow). And finally, the demands of s and t are satisfied, hence there are in total c TA office hours held. $\square$

After the proof of the Claim, one can just apply the algorithm of finding out whether there exists a feasible circulation in the network G or not, but you just had to tell that this is possible, not to prove it, it is a Theorem in the textbook (*this was also explained during the tutorial, you can refer to the textbook to see how this is being done, you can also look at the example on my webpage which explains on a small example how the problem of finding a feasible circulation with lower bounds is equivalent to a problem of finding a maximal flow in some other network*).

(b) Similarly to (a) we construct a flow network but this time instead of having edges $\overrightarrow{(v_j, t)}$ for every u_j, we add nodes w_l for every day l and we add edges $\overrightarrow{(v_j, w_l)}$ of capacity 1 whenever time slot I_j is on day l. We also add all edges $\overrightarrow{(d_l, t)}$ of lower bound d_l and capacity c. Similarly to Claim 2.1 one can show that there exists a feasible circulation in the network if and only if there exists a proper TA scheduling.