Quiz #1 Solution, Computer Science, 308-251A
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Total Time: 40 minutes. Total Marks: 40 (2 questions)
Closed book. **Relax, you have 40 minutes and it's only a quiz.**

Question 1: [Bounding, 20 points]

You are visited by an angel in the middle of the night. The angel says that if you bound the following problem, the world will experience 5 years of peace, love and unity. If you do not bound it, the entire world will suffer and everyone will know that it was you who caused it.

Part (a) (10 points): Let d, q be constants, $d > 1$, $q > 1$. Find $g(n)$ such that the following is true:

$$log\left(\prod_{k=0}^{n}\left(q^{(d^k)}\right)^d\right) = O(g(n)).$$

Solution:

Notice that a log of products can be written as a sum of logs.

$$\begin{aligned} log\left(\prod_{k=0}^{n}\left(q^{(d^k)}\right)^d\right) &= \sum_{k=0}^{n} log\left(\left(q^{(d^k)}\right)^d\right) \\ &= \sum_{k=0}^{n} d \cdot log\left(q^{(d^k)}\right) \\ &= d \cdot \sum_{k=0}^{n} d^k \cdot log\ q \\ &= d\ log\ q \cdot \sum_{k=0}^{n} d^k \end{aligned}$$

Or, you could go this way. Notice that a log of products can be written as a sum of logs.

$$\begin{aligned} log\left(\prod_{k=0}^{n}\left(q^{(d^k)}\right)^d\right) &= log\left(\prod_{k=0}^{n}\left(q^{(d^{k+1})}\right)\right) \\ &= \sum_{k=0}^{n}\left(d^{k+1} \cdot log\ q\right) \\ &= \sum_{k=0}^{n}\left(d \cdot d^k \cdot log\ q\right) \\ &= d\ log\ q \cdot \sum_{k=0}^{n} d^k \end{aligned}$$

$\sum_{k=0}^{n} d^k$ is a geometric series, and since $d > 1$, use the value

$$\sum_{k=0}^{n}(d^k) = \frac{d^{n+1} - 1}{d - 1}.$$

Let's use base q in the logarithm.

$$d \cdot \sum_{k=0}^{n} d^k = (d)\,\frac{d^{n+1}-1}{d-1} = \frac{d}{d-1}\left(d^{n+1}-1\right)$$

Find a $g(n)$ to give a tight upper bound. Find a $g(n)$ so the following is true:

$$\frac{d}{d-1}\left(d^{n+1}-1\right) \le c \cdot g(n)$$

So $\frac{d}{d-1} \le 2$, for any constant $d > 1$.
Let $g(n) = d^{n+1}$.

$$\frac{d}{d-1}\left(d^{n+1}-1\right) \le c \cdot d^{n+1}$$
$$= \frac{d}{d-1} - \frac{d}{(d-1)d^{n+1}} \le c$$

So any c s.t. $c \ge \frac{d}{d-1}$ will do.
Therefore,

$$log\left(\prod_{k=0}^{n}\left(q^{(d^k)}\right)^d\right) = O(d^{n+1}).$$

Part (b) (10 points): Let d, q be constants, $d < 1$, $q > 1$. Find $g(n)$ such that the following is true:

$$log\left(\prod_{k=0}^{n}\left(q^{(d^k)}\right)^d\right) = O(g(n)).$$

Correction as stated at the beginning of the mid-term: $0 \le d < 1$

Solution A:

As shown above,

$$log\left(\prod_{k=0}^{n}\left(q^{(d^k)}\right)^d\right) = d\ log\ q \cdot \sum_{k=0}^{n} d^k$$

Since $0 \le d < 1$, we can bound this using the infinite decreasing geometric series. Let the base of the logarithm be q.

$$d \cdot \sum_{k=0}^{n} d^k \le d \cdot \sum_{k=0}^{\infty} d^k = (d)\,\frac{1}{1-d}$$
$$= \frac{d}{1-d}$$

Let $g(n) = O(1)$. Then,

$$log\left(\prod_{k=0}^{n}\left(q^{(d^k)}\right)^d\right) \leq c.$$

$$\frac{d}{1-d} \leq c.$$

Since d is a constant, $d/(1-d)$ is a constant and we are finished.
Solution B:

As shown above,

$$log\left(\prod_{k=0}^{n}\left(q^{(d^k)}\right)^d\right) = \frac{d\ log\ q}{d-1}\left(d^{n+1}-1\right)$$

Since $0 \leq d < 1$, $-1 \leq (d^{n+1}-1) < 0$ and $\frac{d}{d-1} < 0$. We can see that:

$$\frac{d\ log\ q}{d-1}\left(d^{n+1}-1\right) \leq 0, \quad log\ q \leq 0$$

$$\frac{d\ log\ q}{d-1}\left(d^{n+1}-1\right) \leq \frac{d\ log\ q}{d-1}\cdot(-1), \quad log\ q > 0$$

It is therefore true that

$$\frac{d\ log\ q}{d-1}\left(d^{n+1}-1\right) \leq \left|\frac{d\ log\ q}{d-1}\right|$$

Which gives us $g(n) = 1$ and $c = |\frac{d\ log\ q}{1-d}|$. So,

$$log\left(\prod_{k=0}^{n}\left(q^{(d^k)}\right)^d\right) = O(1).$$

Question 2: [Recurrences, 20 points]

Part (a) (10 points): Show that the following recurrence is $\Omega(n^{r+1})$:

$$T(n) = T(n-1) + \Theta(n^r).$$

Solution:

Write the recurrence as a summation. Imagine the recurrence tree. At each level i, there is $(n-i)^r$ work being done. There are n levels in total, because the input size decreased by one at each recurse.

$$\begin{aligned} T(n) &= n^r + (n-1)^r + (n-2)^r + (n-3)^r + ... + 1^r \\ &= \sum_{k=1}^{n} k^r \end{aligned}$$

Now this is exactly the same as a question from the assignment. First use splitting of sums.

$$\begin{aligned}
\sum_{k=1}^{n} k^r &\geq \sum_{k=1}^{n/2} 0 + \sum_{k=n/2+1}^{n} (k/2)^r \\
&= (n/2)(n/2)^r \\
&= (n/2)^{r+1} \\
&= \frac{n^{r+1}}{2^{r+1}}
\end{aligned}$$

This gives us a $c = \frac{1}{2^{r+1}}$ and $n_0 = 0$ for which the following is true:

$$\sum_{k=1}^{n} k^r \geq c \cdot n^{r+1}, \ \forall n \geq n_0$$

Therefore, $T(n) = T(n-1) + \Theta(n^r) = \Omega(n^{r+1})$.

Part (b) (10 points): Show that the following recurrence is $\Omega(2^n)$:

$$T(n) = 2 \cdot T(n-1) + \Theta(n^r).$$

Solution:

Again, imagine the recurrence tree. This time there are 2^i vertices at each ith level, each doing $(n-i)^r$ work.

$$\begin{aligned}
T(n) &= n^r + 2 \cdot T(n-1) \\
&= n^r + 2((n-1)^r + 2 \cdot T(n-2)) \\
&= n^r + 2 \cdot (n-1)^r + 4 \cdot T(n-2) \\
&= n^r + 2 \cdot (n-1)^r + 4 \cdot (n-2)^r + 8 \cdot (n-3)^r + ... + 2^{n-1} \cdot 1^r \\
&= \sum_{k=0}^{n-1} 2^k \cdot (n-k)^r
\end{aligned}$$

Since $(n-k)^r \geq 1$, we can bound the summation by bounding terms:

$$\begin{aligned}
\sum_{k=0}^{n-1} 2^k \cdot (n-k)^r &\geq \sum_{k=0}^{n-1} 2^k \\
&= 2^n - 1
\end{aligned}$$

Note that the $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ is the total number of nodes in a complete binary tree with n leaves.

We now must show that the following is true:

$$\begin{aligned}
2^n - 1 &\geq c \cdot 2^n \\
\frac{2^n - 1}{2^n} &\geq c \\
1 - \frac{1}{2^n} &\geq c
\end{aligned}$$

This is true when $c = 1/2$ and $n_0 = 1$, because $1/2^n \leq 1/2,\ \forall n \geq 1$. Therefore $T(n) = 2 \cdot T(n-1) + \Theta(n^r) = \Omega(2^n)$.

Equations:

$$H_n = 1 + 1/2 + 1/3 + 1/4 + \cdots + 1/n = ln\ n + O(1)$$

$$1 + x \leq e^x \leq 1 + x + x^2,\ \text{when } |x| \leq 1$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, |x| < 1$$

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n+(1/12n)}$$

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}, x \neq 1$$