COMP 760 - Winter 2017 - Assignment 2

Due: March 7th, 2017

**General rules:** In solving these you may consult with each other.

1. (a) Prove that there exists a universal constant $K$ such that for every function $g : \{0, 1\}^n \to \mathbb{R}$, we have
   \[
   \sum_{S \neq \emptyset} \frac{1}{|S|} \hat{g}(S)^2 \leq \frac{K \|g\|_2^2}{1 + \log(\|g\|_2/\|g\|_1)}.
   \]
   (b) Conclude that for $f : \{0, 1\}^n \to \{0, 1\}$, we have
   \[
   \text{Var}[f] \leq K \sum_i \frac{I_i(f)}{1 + \frac{1}{2} \log I_i(f)}.
   \]
   (c) Conclude the KKL inequality from the above inequality.

2. For a function $f : \{-1, 1\}^n \to [-1, 1]$, define $\partial_i f(x) = \frac{f(x) - f(x \oplus e_i)}{2}$, and let $I_i(f) = \|\partial_i f\|_1$. Let $\Delta f(x) = \sum_i |\partial_i f(x)|$. Prove that if $f$ is of degree $d$, then
   \[
   I_f := \sum_i I_i(f) \leq \|\Delta f\|_\infty \leq 2d^2.
   \]
   Hint: Use Markov brothers’ inequality.

3. An $\oplus$-decision tree for a function $f : \{0, 1\}^n \to \{0, 1\}$ is a decision tree where every internal node is labeled with a subset $S \subseteq \{1, \ldots, n\}$; On an input $x$, we traverse a path from the root to a leaf that is labeled with $f(x)$ where at a node labeled with $S$, we branch according to the value of $\oplus_{i \in S} x_i$. Prove that if $f$ has an $\oplus$-decision tree with $s$ leaves, then $\|\hat{f}\|_1 := \sum_S |\hat{f}(S)| \leq s$.

4. Suppose that $\|\hat{f}\|_1 = M$ for a function $f : \mathbb{Z}_2^n \to \{-1, 1\}$. Let $\alpha = |\hat{f}(S)|$ and $\beta = |\hat{f}(T)|$ denote the two largest Fourier coefficients (in absolute value).
   (a) Denoting $R = S \Delta T$, prove that either
      \[
      |\hat{f}|_{|R=1} \parallel_1 \leq M - \alpha \leq M - \frac{1}{M}, \quad |\hat{f}|_{|R=-1} \parallel_1 \leq M - \beta,$
   
      or
      \[
      |\hat{f}|_{|R=1} \parallel_1 \leq M - \alpha \leq M - \frac{1}{M}, \quad |\hat{f}|_{|R=-1} \parallel_1 \leq M - \beta.
      
      (Here $f|_{|x=1}$ means the restriction of $f$ to the subspace $\{x \in \mathbb{Z}_2^n, \chi_R(x) = 1\} \cong \mathbb{Z}_2^{n-1}$).
   (b) Conclude that if $f : \{0, 1\}^n \to \{-1, 1\}$ is a boolean function with $\|\hat{f}\|_1 = M$, then $f$ can be computed by an $\oplus$-decision tree with at most $n M^2$ leaves.