COMP 760 - Winter 2017 - Assignment 1

Due: Feb 7th, 2017

General rules: In solving these you may consult with each other.

1. Consider a graph $G = (V, E)$ and a set $A \subseteq \{0,1\}^n$ satisfying $\max_{S \neq \emptyset} |\widehat{A}(S)| \leq \delta$, and let $\alpha := \widehat{A}(\emptyset)$. Let $\{x_v : v \in V(H)\}$ be independent random variables taking values in $\{0,1\}^n$ uniformly at random for some number n. Show that
$$\left| \mathbb{E}\left[\prod_{uv \in E} A(x_u + x_v)\right] - \alpha^{|E(G)|} \right| = o_{\delta \to 0}(1),$$
where the bound does not depend on n (it depends only on δ and G).

2. This exercise shows that the Fourier uniformity (i.e. having small non-principal Fourier coefficients) is not sufficient to show that a set behaves similar to a random set in terms of the density of all linear structures. We shall construct a set A_n which is very uniform, but the density of certain structure in A_n is different from a random subset with the same density as A_n.
   (a) Show that the set
   $$A_n = \left\{ x \in \mathbb{Z}_2^{2n} \ : \ \sum_{i=1}^{n} x_{2i-1}x_{2i} \equiv 0 \mod 2 \right\}$$
   satisfies $\max_{S \neq \emptyset} |\widehat{A_n}(S)| = o_{n \to \infty}(1)$.
   (b) Show that $\lim_{n \to \infty} |\widehat{A_n}(\emptyset)| = \frac{1}{2}$.
   (c) Show that
   $$\lim_{n \to \infty} \left| \mathbb{E}\left[\prod_{1 \leq u < v < w \leq 6} 1_{A_n}(x_u + x_v + x_w)\right] - 2^{-\binom{6}{3}} \right| \neq 0,$$
   where $x_1, \ldots, x_6$ are independent random variables taking values independently in $\mathbb{Z}_2^n$.

3. Consider a decision tree computing a Boolean function $f : \{-1,1\}^n \to \{-1,1\}$. For an $x \in \{-1,1\}^n$ and $i \in \{1, \ldots, n\}$ define $R_i(x) = x_i$ if the variable x_i is queried by the decision tree while computing $f(x)$, and define $R_i(x) = 0$ otherwise. Prove that for every i, we have $\widehat{f}(\{i\}) = \mathbb{E}f(x)R_i(x)$, and use this to show that if f is monotone then
$$I[f] \leq \sqrt{h},$$
where h is the height of the decision tree. Can you prove the stronger bound that $I[f] \leq \sqrt{\log_2 s}$ where s is the number of leaves of the tree?

4. Use Marcinkiewicz-Zygmund inequality (See wikipedia) to prove the following: Let $p > 1$ and let $f : \{-1,1\}^n \to \{-1,1\}$ be a Boolean function satisfying $\sum |\widehat{f}(S)| = \lambda$. There exists a function $f : \{-1,1\}^n \to \mathbb{R}$ such that g has at most $O(p/\epsilon^2)$ non-zero Fourier coefficients and it approximates f in the L_p norm:
$$\|f - g\|_p \leq \epsilon\lambda.$$