1. Reductions

- **Reductions:** Given families $f_n, g_n : \{0,1\}^n \times \{0,1\}^m \rightarrow \{0,1\}$, we say $\{f_n\} \leq_{cc} \{g_n\}$ if for every $n$, there exists $h_1, h_2 : \{0,1\}^n \rightarrow \{0,1\}^m$ for $\log(m) \leq \log^{O(1)}(n)$ such that $f_n(x,y) = g_m(h_1(x), h_2(y))$. Note that for all the communication complexity classes $C$ that we defined in the previous lecture, if $\{g_n\} \in C$ and $\{f_n\} \leq_{cc} \{g_n\}$, then $\{f_n\} \in C$.

- **Completeness:** A problem $\{g_n\}$ is $C$-complete for a communication complexity class $C$ if and only if
  1. $\{g_n\} \in C$.
  2. $\{f_n\} \leq_{cc} \{g_n\}$ for every $\{f_n\} \in C$.

One can think of $C$-complete problems as the most difficult problem in the class $C$. The next proposition shows that DISJ is the hardness problem in the class CoNP.

**Proposition 1.** The disjointness problem DISJ is CoNP-complete.

*Proof.* First note that DISJ $\in$ CoNP as if $S \cap T \neq \emptyset$ then the oracle can send an $i \in S \cap T$ and Alice and Bob can both verify this. To show the completeness, consider $\{f_n\} \in$ CoNP. Then by definition of CoNP we have $m := C^0(f_n) \leq 2^{\log^c n}$. Consider the following reduction: $h_1(x)$ is the list of the rectangles in the cover that contain $x$, and $h_2(y)$ is the rectangles in the cover that contain $y$. Note that $|h_1(x)|, |h_2(x)| \leq C^0(f) = \log^{O(1)}(n)$. Furthermore

$$f(x,y) = 0 \iff h_1(x) \cap h_2(y) \neq \emptyset.$$ 

\[\square\]

2. Matrix norms

- **Let** $A \in \mathbb{R}^{m \times n}$. The singular values $\sigma_1 \geq \ldots \geq \sigma_{\min(m,n)} \geq 0$ of $A$ are the square roots of the eigenvalues of $AA^T$. The singular decomposition theorem says

$$A = U \Sigma V^T,$$

for unitary matrices $U_{m \times m}$ and $V_{n \times n}$, where $\Sigma_{m \times n}$ is a diagonal matrix with $\sigma_1, \ldots, \sigma_{\min(m,n)}$ on the diagonal.

- **Matrix inner product:** For $A, B \in \mathbb{R}^{m \times n}$, we have $\langle A, B \rangle := \text{tr}(AB^T) = \sum_{i,j} A_{ij}B_{ij}$.

- **Spectral Norm:** $\|A\| = \max_{\|x\|=1} \|Ax\| = \sigma_1 = \|\vec{\sigma}\|_\infty$. 

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• Trace Norm: \( \|A\|_\Sigma = \sum \sigma_i = \|\vec{\sigma}\|_1 \).

• Frobenius Norm: \( \|A\|_F = \sqrt{\sum \sigma_i^2} = \|\vec{\sigma}\|_2 = \sqrt{\sum_{i,j} A_{ij}^2} = \sqrt{\langle A, A \rangle} \).

• Some of the classical inequalities about the \( L_p \) spaces easily extend to the matrix norms:

Exercise 2. Show that

1. \( \sqrt{\text{rank}(A)} \geq \|A\|_\Sigma / \|A\|_F \).
2. \( \langle A, B \rangle \leq \|A\|_F \|B\|_F \).
3. \( \langle A, B \rangle \leq \|A\| \|B\|_\Sigma \).

3. Forster’s theorem

The goal of this section is to prove Forster’s theorem that the inner product function \( \text{IP}_n \) has large sign-rank. We start from the following technical lemma whose proof is not very easy.

Lemma 3 (Forster [For02]). Let \( U \subseteq \mathbb{R}^r \) be a finite set of vectors in general position\(^1\) and suppose \( |U| \geq r \). There exists a non-singular \( A \in \mathbb{R}^{r \times r} \) such that

\[
\sum_{u \in U} \frac{1}{\|Au\|^2} (Au)(Au)^T = \frac{|U|}{r} I_r.
\]

Proof. See Forster’s paper [For02] or David Stuerer’s exposition on this theorem [http://www.cs.princeton.edu/courses/archive/spr08/cos598D/forster.pdf].

Note that every \( X \times Y \) matrix \( R \) with \( \|R\|_\infty \leq 1 \) satisfies \( \|R\|_F \leq \sqrt{|X||Y|} \). Next we will prove the key proposition of Forster.

Proposition 4. Let \( M \in \mathbb{R}^{X \times Y} \). There exists a matrix \( R \) that sign-represents \( M \), \( \text{rank}(R) = \text{rank}_\pm(M) \), \( \|R\|_\infty \leq 1 \), and

\[
\|R\|_F = \sqrt{\frac{|X||Y|}{\text{rank}_\pm(M)}} = \sqrt{\frac{|X||Y|}{\text{rank}(R)}}.
\]

Proof. Obviously there exists a matrix \( Q \) that sign-represents \( M \) and \( \text{rank}(Q) = \text{rank}_\pm(M) =: r \). We can decompose \( Q \) and obtain vectors \( \{u_x\}_{x \in X}, \{v_y\}_{y \in Y} \subseteq \mathbb{R}^r \) such that

\[
Q = [\langle u_x, v_y \rangle]_{x \in X, y \in Y}.
\]

Since \( \text{rank}(Q) = r \), the vectors \( \{u_x\}_{x \in X} \) are in the general position. Since \( \text{rank}_\pm(M) \leq \text{rank}(M) \), we know \( |X| \geq r \), and thus we can apply Lemma 3 to obtain a non-singular \( A \in \mathbb{R}^{r \times r} \) with

\[
\sum_{x \in X} \frac{1}{\|Au_x\|^2} (Au_x)(Au_x)^T = \frac{|X|}{r} I_r.
\]

Define

\[
R = \left[ \frac{\langle Au_x, (A^{-1})^T v_y \rangle}{\|Au_x\| \|(A^{-1})^T v_y\|} \right]_{x \in X, y \in Y} = \left[ \frac{\langle u_x, v_y \rangle}{\|Au_x\| \|(A^{-1})^T v_y\|} \right]_{x \in X, y \in Y}.
\]

\(^1\)General position means that any set of at most \( r \) points in \( U \) are linearly independent
Obviously \( \text{rank}(R) = r \), \( R \) sign-represents \( M \) and also by Cauchy-Schwarz \( \|R\|_{\infty} \leq 1 \). It remains to verify (1). For a fixed \( y \), we have

\[
\sum_{x \in X} R_{xy}^2 = \sum_{x \in X} \frac{\langle Au_x, (A^{-1})^T v_y \rangle^2}{\|Au_x\|^2\|(A^{-1})^T v_y\|^2}
\]

\[
= \sum_{x \in X} \frac{(v_y^T A^{-1})(Au_x)(Au_x)^T((A^{-1})^T v_y)}{\|Au_x\|^2\|(A^{-1})^T v_y\|^2}
\]

\[
= \frac{(v_y^T A^{-1})(|X| I_r)((A^{-1})^T v_y)}{\|(A^{-1})^T v_y\|^2} = |X|/r \cdot \frac{\langle (A^{-1})^T v_y, (A^{-1})^T v_y \rangle}{\|(A^{-1})^T v_y\|^2}
\]

\[
= |X|/r.
\]

Hence

\[
\|R\|_F^2 = \sum_{x,y} R_{x,y}^2 = \frac{|Y||X|}{r},
\]

and this verifies (1). □

**Theorem 5 (Forster [For12]).** For \( M \) be a sign-matrix,

\[
\text{rank}_{\pm}(M) \geq \frac{\sqrt{|X||Y|}}{\|M\|}.
\]

**Proof.** Let \( R \) be as in Proposition 4 with \( r := \text{rank}(R) = \text{rank}_{\pm}(M) \). Since \( M \) is a sign-matrix, and \( R \) sign-represents \( M \) and satisfies \( \|R\|_{\infty} \leq 1 \), we have

\[
\|R\|_F^2 \leq \sum |R_{x,y}| \leq \langle M, R \rangle \leq \|M\| \|R\|_\Sigma \leq \|M\| \|R\|_F \sqrt{r},
\]

where the last two inequalities are from Exercise 2. Using (1) to replace \( \|R\|_F = \sqrt{|X||Y|}/r \), this simplifies to \( r \geq \frac{\sqrt{|X||Y|}}{\|M\|} \). □

Recall that all the eigenvalues of \( M_{IP_n} \) are of the form \( \pm 2^{n/2} \). Thus \( \|M_{IP_n}\| = 2^{n/2} \). Replacing this in the above theorem proves Forster’s theorem that

\[
\text{rank}_{\pm}(IP_n) \geq \frac{\sqrt{2^n \times 2^n}}{2^{n/2}} = 2^{n/2},
\]

which in particular shows that \( U(IP_n) \geq \frac{n}{2} \). Hence inner product does not belong to the class \( \text{UPP}^{cc} \).

Finally let us mention an extension of Theorem 5 that we will need later in the course.

**Theorem 6 (Razborov-Sherstov [RS10]).** Let \( M \in \mathbb{R}^{X \times Y} \) and \( \gamma \geq 0 \) we have

\[
\text{rank}_{\pm}(M) \geq \frac{\gamma s}{\|M\|\sqrt{s + \gamma h}},
\]

where \( s = |X||Y| \) and \( h = \{|(x, y) : |M(x, y)| < \gamma\}|. \)

**Proof.** Let \( R \) be as in Proposition 4 with \( r := \text{rank}(R) = \text{rank}_{\pm}M \). On the one hand we have

\[
\langle M, R \rangle = \sum_{x,y} M_{xy}R_{xy} \geq \sum_{x,y, |M_{xy}| \geq \gamma} M_{xy}R_{xy} \geq \gamma \left( \sum_{x,y} R_{xy} - h \right) \geq \gamma \left( \sum_{x,y} R_{x,y}^2 - h \right)
\]

\[
= \gamma \|R\|_F^2 - \gamma h.
\]
On the other hand
\[ \langle M, R \rangle \leq \|M\|\|R\| \Sigma \leq \|M\|\|R\|_F \sqrt{r}. \]
Hence
\[ \|M\|\|R\|_F \sqrt{r} \geq \gamma \|R\|^2_F - \gamma h, \]
which using \([X\|Y\]_F = \sqrt{\frac{X^T Y}{r}}\) from (1) simplifies to the desired result. \(\square\)

REFERENCES


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