COMP760, SUMMARY OF LECTURE 5.

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- *Rectangle size lower bounds*: Recall that one can lower-bound $D(f)$ by the logarithm of the size of f's largest fooling set. Another way to lower-bound $D(f)$ is by showing that all the monochromatic rectangles in the communication matrix of f are small: $C^0(f) \geq 2^{2n}/m_0$ and $C^1(f) \geq 2^{2n}/m_1$ where m_0 and m_1 are respectively the sizes of the smallest 0-monochromatic number and 1-monochromatic rectangles.

  More generally let μ be a probability distribution on $\{(x,y) \mid f(x,y) = 1\}$. Then $C^1(f) \geq \frac{1}{\max_R \mu(R)}$ where the maximum is over all 1-monochromatic rectangles R. A similar statement holds for 0's.

- *The inner product function:* We will apply the rectangle size method to $\mathrm{IP}_n : \mathbb{F}_2^n \times \mathbb{F}_2^n \to \mathbb{F}_2$ defined as $\mathrm{IP}_n : (x,y) \mapsto \sum_{i=1}^n x_i y_i =: \langle x, y \rangle$. Note that the number of 0's of IP_n is $\geq 2^{2n-2}$ (why?). Let $S \times T$ be a 0-monochromatic rectangle. Let $S' := \mathrm{span}(S)$ and $T' = \mathrm{span}(T)$. Since
$$\langle a + a', b + b' \rangle = \langle a, b \rangle + \langle a, b' \rangle + \langle a', b \rangle + \langle a', b' \rangle,$$
the rectangle $S' \times T'$ is also 0-monochromatic. Consequently, S' and T' are orthogonal subspaces of $\mathbb{F}_2^n$, and thus $\dim(S') + \dim(T') \leq n$, which shows $|S' \times T'| = |S'||T'| \leq 2^n$. Hence $C^0(\mathrm{IP}_n) \geq 2^{2n-2}/2^n \geq 2^{n-2}$, and $D(\mathrm{IP}_n) \geq \log_2(2^{n-2}) = n - 2$.

Definition 1 (Discrepancy)**.** *Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ and let μ be a probability distribution on $\{0,1\}^n \times \{0,1\}^n$. For a rectangle R define*
$$\mathrm{Disc}_\mu(R, f) = |\mu(R \cap f^{-1}(0)) - \mu(R \cap f^{-1}(1))|.$$
Let
$$\mathrm{Disc}_\mu(f) = \max_R |\mu(R \cap f^{-1}(0)) - \mu(R \cap f^{-1}(1))|,$$
and
$$\mathrm{Disc}(f) = \min_\mu \mathrm{Disc}_\mu(f).$$

- If we change the range to ± 1 (i.e. $f : \{0,1\}^n \times \{0,1\}^n \to \{-1,1\}$), the definition simply becomes
$$\mathrm{Disc}_\mu(R, f) = \int_R f(x,y) d\mu(x,y).$$
If μ is the uniform measure, then this is known as the cut norm
$$\mathrm{Disc}_\mu(f) = \sup_{R = S \times T} \int_R f(x,y) d(x,y) = \mathbb{E}_{x,y}\left[1_S(x) f(x,y) 1_T(y)\right] =: \|f\|_\square.$$

Theorem 2 (Discrepancy lower-bound)**.** $R^{pub}_{\frac{1}{2}-\epsilon}(f) \geq \log \frac{2\epsilon}{\mathrm{Disc}(f)}$.

Proof. From the previous lecture we know $R^{pub}_{\frac{1}{2}-\epsilon} = \max_\mu D^\mu_{\frac{1}{2}-\epsilon}(f)$. Thus it suffices to show $D^\mu_{\frac{1}{2}-\epsilon}(f) \geq \log \frac{2\epsilon}{\mathrm{Disc}(f)}$. See [KN97, Proposition 3.28] for the proof of this fact. $\square$

- The cut norm and thus the discrepancy with respect to the uniform measure is closely related to the largest eigenvalue (For more details see Section 2 of Lecture 3 in Toni Pitassi's course).

  Proposition 3. *Let $f : X \times X \to \{-1, 1\}$ be a symmetric function (i.e. $f(x, y) = f(y, x)$), and let $\lambda_{\max}$ be the largest eigenvalue in the absolute value of the corresponding matrix M_f. Then*
  $$\mathbb{E}_{x,y}\left[1_S(x) f(x, y) 1_T(y)\right] \leq \frac{1}{|X|^2} |\lambda_{\max}| \sqrt{|S||T|}.$$
  In particular
  $$\|f\|_\square \leq \frac{|\lambda_{\max}|}{|X|}.$$

- It is not difficult to see that the matrix of IP_n is the Hadamard matrix, and hence its eigenvalues are all $\pm 2^{n/2}$. It follows that
  $$\mathrm{Disc}(\mathrm{IP}_n) \geq \mathrm{Disc}_{\mathrm{uniform}}(\mathrm{IP}_n) \leq 2^{-n} 2^{n/2} = 2^{-n/2}.$$
  Consequently
  $$R^{pub}_{1/3}(\mathrm{IP}_n) \geq \Omega(n).$$

References

[KN97] Eyal Kushilevitz and Noam Nisan, *Communication complexity*, Cambridge University Press, Cambridge, 1997. MR 1426129 (98c:68074)

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