COMP760, SUMMARY OF LECTURE 11.

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- **Dual characterization of approximate degree:** Fix $\epsilon > 0$, and let $f : \{0,1\}^n \to \mathbb{R}$ be given, $d = \deg_\epsilon(f) \geq 1$. Then there is a function $\Psi : \{0,1\}^n \to \mathbb{R}$ such that

$$\begin{array}{ll} \widehat{\psi}(S) = 0 & (|S| < d) \\ \sum_{x \in \{0,1\}^n} |\psi(x)| = 1 & \\ \sum_{x \in \{0,1\}^n} \psi(x) f(x) > \epsilon. & \end{array}$$

- **Dual characterization of sign degree:** Let $f : \{0,1\}^n \to \mathbb{R}$ be given. Then $\deg_\epsilon(f) > d$ if and only if there is a distribution μ over $\{0,1\}^n$ with

$$\widehat{f\mu}(S) = \mathbb{E}_\mu[f(x)\chi_S(x)] = 0 \qquad (|S| \leq d).$$

- Let F be the (n, t, f)-pattern matrix. If we plug-in the (n, t, ψ)-pattern matrix Ψ, where ψ is from the dual characterization of the approximate degree, in the lower-bound

$$R_\delta(F) \geq \log \frac{\langle F, \Psi \rangle - 2\delta \|\Psi\|_1}{\|\Psi\| \sqrt{|X||Y|}},$$

  and use the bound from the previous lecture [She09, Theorem 4.3],

$$\|\Psi\| = \max_{S : \widehat{\psi}(S) \neq 0} \sqrt{2^{n+t} \left(\frac{n}{t}\right)^{t-|S|}} |\widehat{\psi}(S)|,$$

  together with the obvious bound

$$|\widehat{\psi}(S)| \leq 2^{-t} \sum_z |\psi(z)|,$$

  we obtain the following theorem:

  Theorem 1 ([She09, Theorem 4.8])**.** *For the (n, t, f)-pattern matrix F, and $\epsilon > 0$ and $\delta < \epsilon/2$, we have*

$$R_\delta(F) \geq \frac{1}{2} \deg_\epsilon(f) \log(n/t) - \log(\frac{1}{\epsilon - 2\delta}).$$

- Let F be the (n, t, f)-pattern matrix. Consider the $\left(n, t, 2^{-n} \left(\frac{n}{t}\right)^{-t} \mu f\right)$-pattern matrix Ψ, where μ is from the dual characterization of the approximate degree. Note that $\Psi(x, y) = F(x, y)\nu(x, y)$ where ν is a probability measure on $X \times Y$, where X and Y respectively correspond to the rows and columns of F. Hence

$$\mathrm{disc}_\nu(F) = \mathrm{disc}_{uniform}(\Psi) \leq \|\Psi\| \sqrt{|X||Y|} = \|\Psi\| \sqrt{2^n \left(\frac{n}{t}\right)^t 2^t}.$$

Again using the bound from the previous lecture [She09, Theorem 4.3],

$$\|\Psi\| = \max \bigcup_{S:\widehat{\mu f}(S)\neq 0} \sqrt{2^{n+t}\left(\frac{n}{t}\right)^{t-|S|}}|\widehat{\mu f}(S)|,$$

together with

$$|\widehat{\mu f}(S)| \leq 2^{-t}\sum_{z}|\mu(z)| \leq 2^{-t},$$

we obtain the following theorem:

Theorem 2 ([She09, Theorem 4.13])**.** *For the (n,t,f)-pattern matrix F,*

$$\mathrm{disc}(F) \leq \left(\frac{n}{t}\right)^{-\deg_{\pm}(f)/2}.$$

- It is well-known that he Minsky-Papert function

$$\mathrm{MP}_m(x) = \wedge_{i=1}^{m}\vee_{j=1}^{4m^2} x_{ij}$$

satisfies $\deg_{\pm}(\mathrm{MP}_m) \geq m$. Since the $(8m^3, 4m^3, \mathrm{MP}_m)$-pattern matrix is a submatrix of $[f(x,y)]_{x,y\in\{0,1\}^{4m^3}}$ where $f(x,y) := \mathrm{MP}_m(x\wedge y) = \wedge_{i=1}^{m}\vee_{j=1}^{4m^2}(x_{ij}\wedge y_{ij})$, we conclude that

$$\mathrm{disc}(f) = \mathrm{disc}(\neg f) \leq 2^{-\Omega(m)},$$

which in particular shows that $f \notin \mathrm{PP}^{cc}$. Since $f \in Pi_2^{cc}$ and $\neg f \in \Sigma_2^{cc}$ we conclude

Theorem 3. *We have*

$$\Pi_2^{cc}, \Sigma_2^{cc} \not\subseteq \mathrm{PP}^{cc}.$$

Recall from the assignment 2 that

$$\Pi_1^{cc}, \Sigma_1^{cc} \subseteq \mathrm{PP}^{cc}.$$

References

[She09] Alexander A. Sherstov, *Lower bounds in communication complexity and learning theory via analytic methods*, Ph.D. thesis, The University of Texas at Austin, 8 2009.

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