COMP760, SUMMARY OF LECTURE 10.

HAMED HATAMI

- Definition of pattern matrices (n, t, f): [She09, Definition 4.1].
- The singular values of a pattern matrix (n, t, f): [She09, Theorem 4.3].

$$\bigcup_{S:\widehat{f}(S)\neq 0} \left\{ \sqrt{2^{n+t}\left(\frac{n}{t}\right)^{t-|S|}}|\widehat{f}(S)| \ : \ \text{repeated } \left(\tfrac{n}{t}\right)^{|S|} \text{ times} \right\}.$$

- Generalized discrepancy method:

Theorem 1. *For every $f, h : X \times Y \to \{-1, 1\}$, and every probability distribution μ on $X \times Y$, we have*

$$D^{\mu}_{\epsilon}(f) \geq \log \frac{\mathbb{E}_{\mu} fh - 2\epsilon}{\mathrm{disc}_{\mu}(h)}.$$

Proof. Let $P : X \times Y \to \{-1, 1\}$ be a deterministic protocol with communication cost $c = D^{\mu}_{\epsilon}(f)$ and

$$\Pr_{\mu}[P(x, y) \neq f(x, y)] \leq \epsilon.$$

Equivalently $\mathbb{E}[P(x, y)f(x, y)] \geq 1 - 2\epsilon$. Then

$$\mathbb{E}_{\mu}[Ph] = \mathbb{E}_{\mu}\left[fh1_{[P=f]} - fh1_{[P\neq f]}\right] \geq \mathbb{E}_{\mu} fh - 2\Pr_{\mu}[P \neq f] = \mathbb{E}_{\mu} fh - 2\epsilon.$$

On the other-hand the original discrepancy method shows

$$D^{\mu}_{\delta}(h) \geq \log_2 \frac{1 - 2\delta}{\mathrm{disc}_{\mu}(h)}.$$

Replacing $\delta := \Pr_{\mu}[P \neq h]$ (equivalently $1 - 2\delta = \mathbb{E}_{\mu}[Ph]$) shows

$$c \geq \log \frac{\mathbb{E}_{\mu}[Ph]}{\mathrm{disc}_{\mu}(h)} \geq \log \frac{\mathbb{E}_{\mu} fh - 2\epsilon}{\mathrm{disc}_{\mu}(h)},$$

as desired □

- Combining this with the spectral method that we saw earlier we get:

Theorem 2. *For $f : X \times Y \to \{-1, 1\}$, we have*

$$R_{\epsilon}(f) \geq \log \sup_{\Psi \in \mathbb{R}^{X\times Y}} \frac{\langle M_f, \Psi\rangle - 2\epsilon\|\Psi\|_1}{\|\Psi\|\sqrt{|X||Y|}}.$$

Proof. We can normalize and assume $\|\Psi\|_1 = 1$. Then the correspondense $\Psi(x, y) = h(x, y)\mu(x, y)$ with the previous theorem, and the bound $\mathrm{disc}_{uniform}(\Psi) \leq \|\Psi\|\sqrt{|X||Y|}$ completes the proof. □

References

[She09] Alexander A. Sherstov, *Lower bounds in communication complexity and learning theory via analytic methods*, Ph.D. thesis, The University of Texas at Austin, 8 2009.

School of Computer Science, McGill University, Montréal, Canada
E-mail address: hatami@cs.mcgill.ca