In the previous lecture, we introduced the Noise Stability of boolean functions and stated theorem 2.22 about noise stability of the Majority Function. We start this lecture by proving the corresponding theorem.

1. Noise Stability in Gaussian Space

**Theorem 1.1** (Noise Stability of Majority Function). Let \( \text{Maj}_n : (\mathbb{R}^n, \gamma_n) \rightarrow \{0, 1\} \)

\[
\text{Maj}_n : x \rightarrow \begin{cases} 
1 & \sum x_i \geq 0 \\
0 & \sum x_i < 0
\end{cases}
\]

, then we have

\[
S_\rho(\text{Maj}_n) = \frac{1}{4} + \frac{\arccos(\rho)}{2\pi}
\]

**Proof.**

\[
S_\rho(\text{Maj}_n) = \mathbb{E}[\sum x_i \geq 0 | \sum y_i \geq 0] = \mathbb{E}[\rho \sum y_i + \sqrt{1 - \rho^2} \sum g_i \geq 0 | \sum y_i \geq 0]
\]

where \( y_i \)'s and \( g_i \)'s are i.i.d. Gaussians. \( \sum y_i \) has the same distribution as \( \sqrt{n}h \), where \( h \) is a Gaussian in \( \mathbb{R} \). Similarly, \( \sum g_i \) has distribution the same as \( \sqrt{n}h' \). Therefore, the expected value is equal to:

\[
\mathbb{E}[\rho h + \sqrt{1 - \rho^2} h' \geq 0 | h \geq 0] = \frac{1}{4} + \frac{\arccos \rho}{2\pi}
\]

□

**Definition 1.2** (Gaussian Rearrangement). Given \( A \subset \mathbb{R}^n \) its Gaussian Rearrangement \( A^* \) is defined to be the interval \((t, \infty)\) with \( \gamma_1(t, \infty) = \gamma_n(A) \).

Recall that \( \gamma_i \) is the Gaussian measure on \( \mathbb{R}^i \).

**Theorem 1.3** (Borrell 83). Let \( A, B \subseteq \mathbb{R}^n \). Then for any \( 0 \leq \rho \leq 1 \) and \( q \geq 1 \) we have:

\[
\mathbb{E}(U_\rho A)^q B \leq \mathbb{E}(U_\rho A^*)^q B^*
\]

In particular,

\[
S_\rho(A) = \mathbb{E}U_\rho A \leq \mathbb{E}A^*U_\rho A^* = S_\rho(A^*)
\]

Hence, \( \gamma_n(A) = \frac{1}{2} \) then \( S_\rho(A) \leq S_\rho(\text{Maj}_n) = \frac{1}{4} + \frac{\arccos \rho}{2\pi} \).

These notes are scribed by Athena K. Moghaddam.
2. Invariance Principle

**Theorem 2.1** (Invariance Principal I). Let $Q(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} \alpha_S \prod_{i \in S} x_i$ satisfies:

(1) $\deg(Q) \leq d$

(2) $\sum_{|S| > 0} \alpha_S^2 = 1$

(3) $I_i := \sum_{S: i \in S} \alpha_S^2 \leq \tau \ \forall i: 1, \ldots, n$

Then:

$$\sup_{t} |\text{prob}[Q(\varepsilon_1, \ldots, \varepsilon_n) \leq t] - \text{prob}[Q(g_1, \ldots, g_n) \leq t]| \leq O(dr^{1/2})$$

Where $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. $\pm 1$ uniform random variables and $g_1, \ldots, g_n$ are i.i.d. Gaussians.

**Definition 2.2** (Rademacher Random Variable). A uniform $\pm 1$ random variable is called a rademacher random variable.

**Theorem 2.3** (Invariance Principal II).

$$|\mathbb{E}[\Psi(Q(\varepsilon_1, \ldots, \varepsilon_n))] - \mathbb{E}[\Psi(Q(g_1, \ldots, g_n))]| \leq O(d^{9/4}B\tau)$$

for all $\Psi: \mathbb{R} \rightarrow \mathbb{R}$ in $C^4$ (four times differentiable) with $|\Psi^{(4)}(t)| < B$ for all $t$.

Remark that if we could take $\Psi: x \rightarrow \begin{cases} |x| & x \leq t \\ 0 & \text{otherwise} \end{cases}$ then theorem II would imply theorem I.

One instead has to approximate functions with bounded fourth derivatives.

**Proof.** Let $Z_i = Q(g_1, \ldots, g_i, \varepsilon_{i+1}, \ldots, \varepsilon_n)$. We claim that $|\mathbb{E}[\Psi(Z_{i-1}) - \mathbb{E}[\Psi(Z_i)]| \leq O(B^{9}I_{i}^{2})$. First we show that the theorem can be extracted from this claim. Indeed,

$$|\mathbb{E}[\Psi(Z_0)] - \mathbb{E}[\Psi(Z_n)]| \leq \sum_{i=1}^{n} |\mathbb{E}[\Psi((Z_{i-1}) - \mathbb{E}[\Psi(Z_i)]|$$

$$\leq O(B^{9}d^{4}) \sum_{i=1}^{n} I_{i}^{2} = O(B^{9}d^{4})$$

$$\max_{i} I_{i} \leq O(B^{9}d^{4}) \sum_{i=1}^{n} I_{i} = O(B^{9}d^{4}) \sum_{|S| > 0} |S|/alpha_S^{2} \leq O(dB^{9}d^{4}) \sum_{|S| > 0} \alpha_S^{2} = O(\tau B^{9}d^{4})$$

To prove the claim $Q(x_1, \ldots, x_n) = \sum_{S \subseteq \mathbb{R}} \alpha_S \prod_{j \in S} x_j + \sum_{S \subseteq \mathbb{R}} \alpha_S \prod_{j \in S} x_j = r(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) + s(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, let $R = r(g_1, \ldots, g_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_n)$ and $S = s(g_1, \ldots, g_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_n)$. We have $Z_{i-1} = R + \varepsilon_i S$ and $Z_i = R + g_i S$. Now using Taylor’s theorem:

$$|\mathbb{E}[\Psi(Z_{i-1}) - \mathbb{E}[\Psi(Z_i)]| \leq$$

$$|\mathbb{E}[\Psi(R) + \varepsilon_i S \Psi'(R) + \frac{(\varepsilon_i S)^2}{2} \Psi''(R) + \frac{(\varepsilon_i S)^3}{6} \Psi^{(3)}(R) + E_1] - \mathbb{E}[\Psi(R) - g_i S \Psi'(R) - \frac{(g_i S)^2}{2} \Psi''(R) - \frac{(g_i S)^3}{6} \Psi^{(3)}(R) - E_2]|$$

Where $|E_1| \leq \frac{|\Psi^{(4)}(t)| |\varepsilon_i|^4 S^4}{24} \leq \frac{B(\varepsilon_i S)^4}{24}$ for some $\xi$ between $R$ and $R + \varepsilon_i S$. Similarly, $|E_2| \leq \frac{B(g_i S)^4}{24}$.

All terms get canceled except the error terms $E_1$ and $E_2$. So the expression is bounded by:

$$\mathbb{E}|\frac{B(\varepsilon_i S)^4}{24}| + \mathbb{E}|\frac{B(g_i S)^4}{24}| \leq \frac{B}{24} E S^4 + \frac{3B}{24} E S^4 B E S^4$$
by Hypercontractivity

$$\leq \frac{B6^d}{6}(\mathbb{E}S^2)^2 = \frac{B6^d}{6} \sum_{S \in \mathcal{S}} \alpha_S^2 = \frac{B6^d}{6} I_i^2$$

\[ \square \]

Lecture 15

In previous lectures, we claimed that Majority Function is the stablest in Gaussian space. Now, we are going to prove this fact using the properties of Threshold Function.

**Definition 2.4.** $T_\rho Q = \sum \rho^{|S|} \alpha_S \prod_{i \in S} x_i$

**Definition 2.5** (Threshold Function). For any $\mu \in [-1, 1]$, the function $\text{Thr}(\mu) : (\mathbb{R}, \gamma_1) \rightarrow \{-1, 1\}$ is defined as:

$$\text{Thr}(\mu) : x \rightarrow \begin{cases} 1 & x \geq t_0 \\ -1 & x < t_0 \end{cases}$$

with $\mathbb{E}\text{Thr}(\mu) = \mu$.

**Theorem 2.6** (Majority is stablest in Gaussian space). Let $f : (\mathbb{R}^n, \gamma_n) \rightarrow [-1, 1]$ with $\mathbb{E} f = \mu$. Then:

$$S_\rho(f) \leq S_\rho(\text{Thr}(\mu))$$

**Theorem 2.7** (Majority is stablest in discrete setting). Let $f : \{0, 1\}^n \rightarrow [-1, 1]$ and $I_i(f) = \sum_{S \ni i} |\hat{f}(S)|^2 \leq \tau$ for every $i$. Then for every $0 \leq \rho \leq 1$, $S_\rho(f) \leq S_\rho(\text{Thr}(\mu)) + O(\rho \log \log \frac{1}{\rho \log \frac{1}{\tau}})$ where $\mu = \mathbb{E} f$

**Proof.** Express $f = \sum \hat{f}(S) \chi_S$. Let $Q(x_1, ..., x_n) = \sum_{S \ni [n]} \hat{f}(S) \prod_{i \in S} x_i$. Therefore, $f(x_1, ..., x_n) = Q(\varepsilon_1, ..., \varepsilon_n)$ where $\varepsilon_i = (-1)^{x_i}$. Let $(g_1, ..., g_n)$ be an i.i.d. Gaussian. We have

$$S_\rho(f) = \sum \rho^{|S|} |\hat{f}(S)|^2 = S_\rho(Q(g_1, ..., g_n))$$

We would like to apply invariance principal to replace rademachers with Gaussians. However, since the degree of $Q$ can be as large as $n$, we cannot apply invariance directly to $Q$. Instead, we apply a smoothed version of the theorem, which can be applied on $T_\beta Q$ for $\beta < 1$. Let $\rho = \rho' / \beta^2$ where $\beta < 1$ is very close to 1. $(0 < 1 - \beta << 1 - \rho)$ to be determined later.

$$S_\rho(f) = \sum \rho^{|S|} |\hat{f}(S)|^2 = \sum (\rho' \beta^2)^{|S|} |\hat{f}(S)|^2 = S_\rho(T_\beta Q(g_1, ..., g_n)).$$

Now using the smoothed invariance, $T_\beta Q(g_1, ..., g_n)$ is close in distribution to $T_\beta Q(\varepsilon_1, ..., \varepsilon_n)$ and hence it cannot be far from being in $[-1, 1]$. To make this precise we define function $\xi$ as follows:

$$\xi : t \rightarrow \begin{cases} 0 & |t| \leq 1 \\ (|t| - 1)^2 & |t| > 1 \end{cases}$$
Note that $\xi$ measures the $L_2$-distance of $t$ from its truncated value in $[-1, 1]$. By invariance principle of random variables $R = T_\beta Q(\varepsilon_1, \ldots, \varepsilon_n)$ and $S = T_\beta Q(g_1, \ldots, g_n)$ satisfy $|E\xi(R) - E\xi(S)| \leq \tau^{\Omega(1-\beta)}$. Let $S'$ be the truncation of $S$ to the interval $[-1, 1]$:

$$S' = \begin{cases} 
S & |S| \leq 1 \\
1 & S > 1 \\
-1 & S < -1 
\end{cases}$$

By assumption, $Q(\varepsilon_1, \ldots, \varepsilon_n) \in [-1, 1]$ and since $T_\beta$ is an an operator, $T_\beta Q(\varepsilon_1, \ldots, \varepsilon_n) \in [-1, 1]$ and hence $\xi(R) = 0$.

Thus,

$$E|\xi(S)| = E(S - S')^2 \leq \tau^{\Omega(1-\beta)}$$

$$\Rightarrow |S_\rho(S) - S_\rho(S')| = |ESU_\rho S - ES'U_\rho S'|$$

$$\leq |ESU_\rho S - ESU_\rho S'| + |ES'U_\rho S - ES'U_\rho S'|$$

$$\leq \|S - S'|_2\|U_\rho S\|_2 + \|S'|_2\|U_\rho (S - S')\|_2$$

$$\leq \|S - S'|_2\|S\|_2 + \|S'|_2\|S - S'|_2 \leq \tau^{\Omega(1-\beta)}.$$}

By Borrell’s theorem, $S_\rho(S') \leq S_\rho(Thr^{\mu'})$ where $\mu' = ES'$. Now, we just have to show that $\mu = \mu'$:

$$|\mu - \mu'| = |E(S - S')| \leq \|S - S'|_2 \leq \tau^{\Omega(1-\beta)}$$

$$\Rightarrow |S_\rho(Thr^{\mu}) - S_\rho(Thr^{\mu'})| \leq O\left(\frac{1 - \beta}{1 - \rho}\right)$$

$$\Rightarrow S_\rho(f) = S_\rho(Thr^{(\mu)}) + O\left(\tau^{\Omega(1-\beta)} + \frac{1 - \beta}{1 - \rho}\right)$$

and by optimizing the last expression over $\beta$ the result yields to the theorem claim. \qed

3. Applications of “Majority is Stablest” Theorem

Definition 3.1 (Condorcet Method for Ranking 3 Candidates). In an election with $n$ voters and 3 candidates, $A$, $B$ and $C$, each voter submits 3 bits representing his preferences. The first bit indicates whether he prefers $A$ to $B$; the second one shows his preference between $B$ and $C$ and the third one shows the same fact over $C$ and $A$. These preferences are aggregated into 3 strings $x, y, z \in \{-1, 1\}^n$. A boolean function $f : \{-1, 1\}^n \mapsto \{-1, 1\}$ is applied to $x, y$ and $z$ and the aggregated preference is represented by $(f(x), f(y), f(z))$.

Definition 3.2 (Condorcet Paradox). If $f$ is the Majority function we have an irrational outcome, in which all 3 aggregated bits are 1 or all are -1 representing preferences $A < B < C < A$ or $A > B > C > A$.

Definition 3.3. A triple $(a, b, c) \in \{-1, 1\}^3$ is called rational, if it corresponds to a non-cyclic ordering.

Theorem 3.4 (Ken Arrow’s Impossibility Theorem). The only functions $f$ that never give irrational outcomes are dictator functions $f(x) = x_i$ or $f(x) = 1 - x_i$ for some $i$.

Note that every voter has 6 possible rational rankings. Suppose that every voter votes independently at random from the 6 possible choices. Let $x, y, z \in \{-1, 1\}^n$ be the corresponding random string. Note that:

$$1_{[a_1 = a_2 = a_3]} = \frac{1}{4} + \frac{1}{4}a_1a_2 + \frac{1}{4}a_1a_3 + \frac{1}{4}a_2a_3$$
\[ \Pr[(f(x), f(y), f(z))] = 1 - \mathbb{E}1_{f(x)=f(y)=f(z)} = \frac{3}{4} - \frac{1}{4} \mathbb{E}f(x)f(y) - \frac{1}{4} \mathbb{E}f(x)f(z) - \frac{1}{4} \mathbb{E}f(y)f(z) \]

\[ = \frac{3}{4} - \frac{3}{4} \mathbb{E}f(x)f(y) = \frac{3}{4} - \frac{3}{4} \sum \widehat{f(S)} \widehat{f(T)} \mathbb{E} \chi_S(x) \chi_T(y) \]

Now we know that,

\[ \mathbb{E} \chi_S(x) \chi_T(y) = (\prod_{i \in S \cap T} \mathbb{E} x_i y_i)(\prod_{i \in S \setminus T} \mathbb{E} x_i)(\prod_{i \in T \setminus S} \mathbb{E} y_i) \]

Since \( \mathbb{E} y_i = \mathbb{E} x_i = 0 \) and \( \mathbb{E} x_i y_i = \frac{2}{6} - \frac{4}{6} = -\frac{1}{3} \), so \( \mathbb{E} \chi_S(x) \chi_T(y) = \begin{cases} 0 & S \neq T \\ \left(\frac{-1}{3}\right)^{|S|} & S = T \end{cases} \). Hence,

\[ \Pr[(f(x), f(y), f(z)) \text{ is rational}] = \frac{3}{4} + \frac{3}{4} \sum (-1)^{|S|} |\widehat{f(S)}|^2 \leq \frac{3}{4} + \frac{3}{4} \mathbb{S}_4(f) \]

**Corollary 3.5.** If \( f \) satisfies \( I_i(f) = o_n(1) \) and \( \mathbb{E} f = 0 \), then rationality of \( f \leq \frac{3}{4} + \frac{3}{4} \arcsin \frac{1}{3} + o_n(1) \leq 0.9123 + o_n(1) \).