COMP 760 - Fall 2011 - Assignment 2

Due: Nov 2nd, 2011

General rules: In solving this you may consult with each other.

1. This exercise shows that the uniformity is not sufficient to show that a set behaves similar to a random set in terms of the density of all linear structures. We shall construct a set $A_n$ which is very uniform, but the density of certain structure in $A_n$ is different from a random subset with the same density as $A_n$.

(a) Show that the set

$$A_n = \left\{ x \in \mathbb{Z}_2^{2n} : \sum_{i=1}^{n} x_{2i-1} x_{2i} \equiv 0 \mod 2 \right\}$$

is $o_{n \to \infty}(1)$-uniform.

(b) Show that $\lim_{n \to \infty} \frac{|A_n|}{2^n} = \frac{1}{2}$.

(c) Show that

$$\lim_{n \to \infty} \left| \mathbb{E} \left[ \prod_{1 \leq u < v < w \leq 6} 1_{A_n}(x_u + x_v + x_w) \right] - 2^{-\binom{6}{3}} \right| \neq 0,$$

where $x_1, \ldots, x_6$ are independent random variables taking values independently in $\mathbb{Z}_2^n$.

2. Consider a graph $H = (V, E)$ and a finite Abelian group $G$. Suppose that $A \subseteq G$ is $\delta$-uniform and $\alpha := \frac{|A|}{|G|}$. Let $\{x_v : v \in V(H)\}$ be independent random variables taking values in $G$ uniformly at random. Show that

$$\left| \mathbb{E} \left[ \prod_{u \in E} A(x_u + x_v) \right] - \alpha^{|E(H)|} \right| = o_{\delta \to 0}(1).$$

3. Consider a function $f : \{0, 1\}^n \to \{0, 1\}$ and a set $T \subseteq [n]$. Prove that for every $1 < p \leq 2$, and every $k$,

$$\left( \sum_{S \subseteq T, 1 \leq |S| \leq k} |\hat{f}(S)|^2 \right)^{1/2} \leq 2 \left( \frac{1}{\sqrt{p-1}} \right)^k \left( \sum_{S \subseteq T \neq \emptyset} |\hat{f}(S)|^2 \right)^{1/p}.$$

4. Consider a function $f : \{0, 1\}^n \to \{0, 1\}$ and a set $T \subseteq [n]$. For every $x \in \{0, 1\}^T$, let $f_x : \{0, 1\}^{[n] \setminus T} \to [-1, 1]$ denote the function $f_x : y \mapsto f(x, y)$. Prove that if $\text{deg}(f) \leq k$, then for every subset $S \subseteq \mathcal{P}([n] \setminus T)$, we have

$$\mathbb{E}_x \left[ \max_{S \subseteq S} |\hat{f}_x(S)| \right] \leq 3^{2k} \max_{S \subseteq S} \left( \sum_{P \supseteq S} |\hat{f}(P)|^2 \right)^{1/4}.$$
5. Let $C_1, \ldots, C_m$ be $\land$-clauses such that $\sum_{i=1}^m C_i \equiv 1$. Let $k > 0$ be an integer, and $f : \{0, 1\}^n \to \mathbb{R}$ satisfy $f = f_k$. Pick a uniform $y \in \{0, 1\}^n$ and let $C_i$ be the unique clause satisfied by $y$ and $T$ be the set of variables in $C_i$. Prove

$$\mathbb{E}_y \left( \mathbb{E}_{x[T]} [f(y_T; x[n\setminus T])] \right)^2 \leq \sum_{S \subseteq [n]} \Pr [S \cap T \neq \emptyset] |\hat{f}(S)|^2.$$