COMP 760 - Fall 2011 - Assignment 1

Due: October 3rd, 2011

General rules: In solving this you may consult with each other, but you must each find and write your own solution.

1. Recall that by Hölder’s inequality, if $p, q \geq 1$ are conjugate exponents and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are complex numbers, then

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left( \sum_{i=1}^{n} |b_i|^q \right)^{1/q}.$$

Deduce from this, that if $p_1, \ldots, p_n$ are non-negative numbers with $\sum_{i=1}^{n} p_i = 1$, then

$$\left| \sum_{i=1}^{n} a_i b_i p_i \right| \leq \left( \sum_{i=1}^{n} |a_i|^{p_i} \right)^{1/p} \left( \sum_{i=1}^{n} |b_i|^{q_i} \right)^{1/q}.$$

2. Let $X$ be a probability space, and $p, q \geq 1$ be conjugate exponents. Show that for every $f \in L_p(X)$, we have

$$\|f\|_p = \sup_{g, \|g\|_q = 1} |\langle f, g \rangle|.$$

3. Suppose that $(X, \mu)$ is a measure space and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, for $p, q, r \geq 1$. Show that if $f \in L_p(X)$, $g \in L_q(X)$, and $h \in L_r(X)$, then

$$\left| \int f(x) g(x) h(x) d\mu(x) \right| \leq \|f\|_p \|g\|_q \|h\|_r.$$

4. Suppose that $X$ is a measure space and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, for $p, q, r \geq 1$. Show that if $f \in L_p(X)$ and $g \in L_q(X)$, then

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

5. Let $X$ be a probability space. Let $\|T\|_{p \to q}$ denote the operator norm of $T : L_p(X) \to L_q(X)$. In other words

$$\|T\|_{p \to q} := \sup_{f, \|f\|_p = 1} \|Tf\|_q.$$

Recall that the adjoint of $T$ is an operator $T^*$ such that

$$\langle Tf, g \rangle = \langle f, T^* g \rangle,$$

for all $f, g \in L_2(X)$. Prove that for conjugate exponents $p, q \geq 1$, and every linear operator $T : L_2(X) \to L_2(X)$, we have

$$\|T\|_{p \to 2} = \|T^*\|_{2 \to q}.$$
6. Let $G$ be a finite Abelian group, and $H$ be a subgroup of $G$. Prove that 
\[
(H^\perp)^\perp = H.
\]

7. Let $G$ be a finite Abelian group, and $H$ be a subgroup of $G$. Prove that for every $f : G \to \mathbb{C}$, we have 
\[
\mathbb{E}_{x \in H} f(x) = \sum_{a \in H^\perp} \hat{f}(\chi_a).
\]

8. Let $G$ be a finite Abelian group and $f, g : G \to \mathbb{C}$. Show that for every positive integer $m$, 
\[
\|f * g\|_m \leq \|f\|_1 \|g\|_m.
\]

9. Let $G$ be a finite Abelian group and $f, g, h : G \to [-1, 1]$. Show that 
\[
\left| \sum_{a \in G} \hat{f}(a) \hat{g}(a) \hat{h}(a) \right| \leq 3 \left\| \min(\|\hat{f}\|, \|\hat{g}\|, \|\hat{h}\|) \right\|_\infty.
\]

10. Let $f : \mathbb{Z}_2^m \to \mathbb{C}$ satisfy $\|f\|_p \leq \sqrt{p}\|f\|_2$ for all $1 \leq p < \infty$, and let $A \subseteq \mathbb{Z}_2^n$. Show that 
\[
\left\| \max_{a \in A} |f_a| \right\|_2 \leq 5 \sqrt{\ln |A|} \|f\|_2,
\]
where $f_a : x \mapsto f(x + a)$.