COMP 760 - Fall 2011 - Assignment 1

Due: October 3rd, 2011

General rules: In solving this you may consult with each other, but you must each find and write your own solution.

1. Recall that by Hölder's inequality, if $p, q \geq 1$ are conjugate exponents and $a_1, \ldots, a_n, b_1, \ldots, b_n$ are complex numbers, then

$$\left| \sum_{i=1}^{n} a_i b_i \right| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q \right)^{1/q} .$$

   Deduce from this, that if $p_1, \ldots, p_n$ are non-negative numbers with $\sum_{i=1}^{n} p_i = 1$, then

$$\left| \sum_{i=1}^{n} a_i b_i p_i \right| \leq \left(\sum_{i=1}^{n} |a_i|^p p_i \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q p_i \right)^{1/q} .$$

2. Let X be a probability space, and $p, q \geq 1$ be conjugate exponents. Show that for every $f \in L_p(X)$, we have

$$\|f\|_p = \sup_{g : \|g\|_q = 1} |\langle f, g \rangle| .$$

3. Suppose that (X, μ) is a measure space and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, for $p, q, r \geq 1$. Show that if $f \in L_p(X)$, $g \in L_q(X)$, and $h \in L_r(X)$, then

$$\left| \int f(x) g(x) h(x) d\mu(x) \right| \leq \|f\|_p \|g\|_q \|h\|_r .$$

4. Suppose that X is a measure space and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, for $p, q, r \geq 1$. Show that if $f \in L_p(X)$ and $g \in L_q(X)$, then

$$\|fg\|_r \leq \|f\|_p \|g\|_q .$$

5. Let X be a probability space. Let $\|T\|_{p \to q}$ denote the operator norm of $T : L_p(X) \to L_q(X)$. In other words

$$\|T\|_{p \to q} := \sup_{f : \|f\|_p = 1} \|Tf\|_q .$$

   Recall that the adjoint of T is an operator T^* such that

$$\langle Tf, g \rangle = \langle f, T^* g \rangle,$$

   for all $f, g \in L_2(X)$. Prove that for conjugate exponents $p, q \geq 1$, and every linear operator $T : L_2(X) \to L_2(X)$, we have

$$\|T\|_{p \to 2} = \|T^*\|_{2 \to q} .$$

6. Let G be a finite Abelian group, and H be a subgroup of G. Prove that
$$\left(H^{\perp}\right)^{\perp} = H.$$

7. Let G be a finite Abelian group, and H be a subgroup of G. Prove that for every $f : G \to \mathbb{C}$, we have
$$\mathbb{E}_{x \in H} f(x) = \sum_{a \in H^{\perp}} \widehat{f}(\chi_a).$$

8. Let G be a finite Abelian group and $f, g : G \to \mathbb{C}$. Show that for every positive integer m,
$$\|f * g\|_m \leq \|f\|_1 \|g\|_m.$$

9. Let G be a finite Abelian group and $f, g, h : G \to [-1, 1]$. Show that
$$\left| \sum_{a \in G} \widehat{f}(a)\widehat{g}(a)\widehat{h}(a) \right| \leq 3 \left\| \min(|\widehat{f}|, |\widehat{g}|, |\widehat{h}|) \right\|_{\infty}.$$

10. Let $f : \mathbb{Z}_2^n \to \mathbb{C}$ satisfy $\|f\|_p \leq \sqrt{p}\|f\|_2$ for all $1 \leq p < \infty$, and let $A \subseteq \mathbb{Z}_2^n$. Show that
$$\left\| \max_{a \in A} |f_a| \right\|_2 \leq 5\sqrt{\ln |A|}\|f\|_2,$$
where $f_a : x \mapsto f(x + a)$.