1. Consider the following linear program.

\[ \begin{align*}
\text{max} & \quad 3x_1 - 2x_2 - 7x_3 \\
\text{s.t.} & \quad x_1 + x_2 = 5 \\
& \quad x_1 - x_3 \leq 3 \\
& \quad x_2 + 2x_3 \geq 4 \\
& \quad x_2 \leq 0
\end{align*} \]

(a) (10 points) Write the dual of the above linear program.

(b) (10 points) Use solution \((-2, 5, -1)\) to the dual program to show that \(x_1 = 5, x_2 = 0, x_3 = 2\) is an optimal solution for the primal linear program. You are not allowed to use the weak or strong duality theorems; instead you have to deduce the correct upper bound by combining the constraints using the dual solution (as in the proof of the weak duality theorem).

2. (10 points) Use the complementary slackness to show that \(x^*_1 = x^* = 0.5, x^*_2 = x^*_4 = 0, x^*_5 = 2\) is an optimal solution for the following Linear Program:

\[ \begin{align*}
\text{max} & \quad 3.1x_1 + 10x_2 + 8x_3 - 45.2x_4 + 18x_5 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 - x_4 + 2x_5 \leq 5 \\
& \quad 2x_1 - 4x_2 + 1.2x_3 + 2x_4 + 7x_5 \leq 16 \\
& \quad x_1 + x_2 - 3x_3 - x_4 - 10x_5 \leq -20 \\
& \quad 3x_1 + x_2 + 3x_3 + \frac{3}{2}x_4 + \frac{7}{4}x_5 \leq 10 \\
& \quad x_2 + x_3 + 6x_4 + 2x_5 \leq 4.5 \\
& \quad 2x_2 - x_4 + x_5 \leq 2 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{align*} \]

3. Consider a flow network \((G, s, t, \{c_e\}_{e \in E})\), and let \(\mathcal{P}\) denote the set of all \(s - t\)-paths in \(G\).

(a) (10 points) Explain why the following linear program solves the MAX-Flow problem:

\[ \begin{align*}
\text{max} & \quad \sum_{P \in \mathcal{P}} x_P \\
\text{s.t.} & \quad \sum_{P : e \in P} x_P \leq c_e \quad \forall e \in E \\
& \quad x_P \geq 0 \quad \forall P \in \mathcal{P}
\end{align*} \]

(Here \(\sum_{P : e \in P}\) means that the sum is over all paths \(P\) that contain the edge \(e\).)
(b) (10 points) Write the dual of the above linear program.

(c) (10 points) Prove that every \( s-t \)-cut \((A, B)\) provides a feasible solution to the dual linear program of Part (b) such that the value of the dual linear program equals to the capacity of the cut. Thus conclude that the solution to the dual linear program is at most the capacity of the min-cut.

4. (15 Points) Formulate the following problem as a linear program: Let \( G = (V, E) \) be an undirected graph. We want to assign a positive number to every vertex of \( G \) such that the total sum of these numbers is 1, and the largest load of a vertex in the graph is minimized. The load of a vertex is the sum of the number on that vertex and the numbers on its immediate neighbours.

5. We are given two positive numbers \( n \) and \( d \) and a function \( f : \{0, 1, \ldots, n\} \rightarrow \mathbb{R} \). Our goal is to find the best approximation of \( f \) with a polynomial of degree \( d \). More precisely we want to find a polynomial of degree \( d \) that minimizes \( \max_{x \in \{0, \ldots, n\}} |f(x) - p(x)| \).

(a) (15 points) Formulate this problem as a linear program.

(b) (10 points) Write the dual of your linear program.