Lecture 14: Sequential decision making. Markov Decision Processes

- Markov Decision Processes
- Policies and value functions
- Dynamic programming methods for computing value functions
  - Policy evaluation
  - Policy improvement

Sequential Decision Making

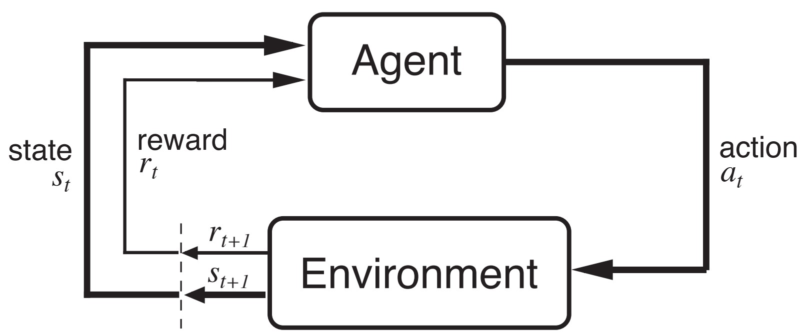
- Decision graphs provide a useful tool for decision making
- If more than one decision has to be taken, reasoning about all of them in general is very expensive
- In bandit problems, the assumption is of *repeated* interaction with an unknown environment over time
- But in a bandit problem, the environment has no "state"
- *Markov Decision Processes (MDPs)* provide a framework for modeling *sequential decision making*, where the environment has different states
- Next class we see what to do if the environment is also unknown

The General Problem: Control Learning

- Robot learning to dock on battery charger
- Choosing actions to optimize factory output
- Playing Backgammon, Go, Poker, ...
- Choosing medical tests and treatments for a patient with a chronic illness
- Conversation
- Portofolio management
- Flying a helicopter
- Queue / router control

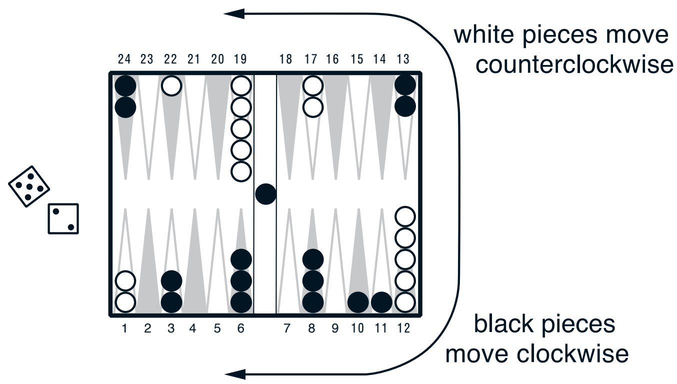
All of these are sequential decision making problems

Reinforcement Learning Problem



- At each discrete time t, the agent (learning system) observes state $s_t \in S$ and chooses action $a_t \in A$
- Then it receives an immediate *reward* r_{t+1} and the state changes to s_{t+1}

Example: Backgammon (Tesauro, 1992-1995)

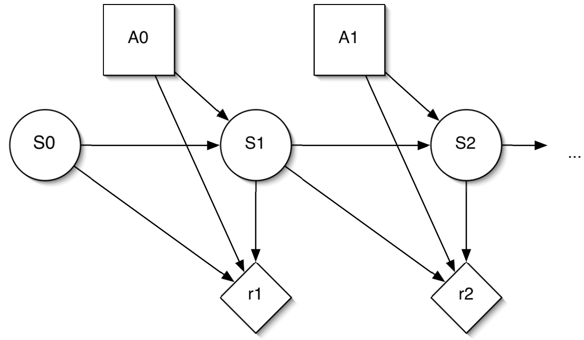


- The states are board positions in which the agent can move
- The actions are the possible moves
- Reward is 0 until the end of the game, when it is ± 1 depending on whether the agent wins or loses

Markov Decision Processes (MDPs)

- Finite set of *states* S (we will lift this later)
- Finite set of *actions* A
- γ = *discount factor* for future rewards (between 0 and 1, usually close to 1). Two possible interpretations:
  - At each time step there is a $1 - \gamma$ chance that the agent dies, and does not receive rewards afterwards
  - Inflation rate: if you receive the same amount of money in a year, it will be worth less
- *Markov assumption:* s_{t+1} and r_{t+1} depend only on s_t and a_t but not on anything that happened before time t

MDPs as Decision Graphs



- The graph may be *infinite*
- But it has a very regular structure!
- At each time slice *the structure and parameters are shared*
- We will exploit this property to get efficient inference

Models for MDPs

- Because of the Markov property, an MDP can be completely described by:
  - *Reward function* $r : S \times A \to \mathbb{R}$
    $r(s, a)$ = the immediate reward if the agent is in state s and takes action a
    This is the *short-term utility* of the action
  - *Transition model* (dynamics): $p : S \times A \times S \to [0, 1]$
    $p(s, a, s')$ = probability of going from s to s' under action a

$$p(s, a, s') = P(s_{t+1} = s' | s_t = s, a_t = a)$$

- These form the *model* of the environment

Planning in MDPs

- The goal of an agent in an MDP is to be rational, i.e., *maximize its expected utility* (respect MEU principle)
- In this case, maximizing the immediate utility (given by the immediate reward) is not sufficient.
  - E.g., the agent might pick an action that gives instant gratification, even if it later makes it "die"
- Hence, the goal is to maximize *long-term utility*, also called *return*
- The return is defined as an additive function of all rewards received by the agent.

Returns

- The *return* R_t for a trajectory, starting from time step t, can be defined depending on the type of task
- *Episodic tasks* (e.g. games, trips through a maze etc)

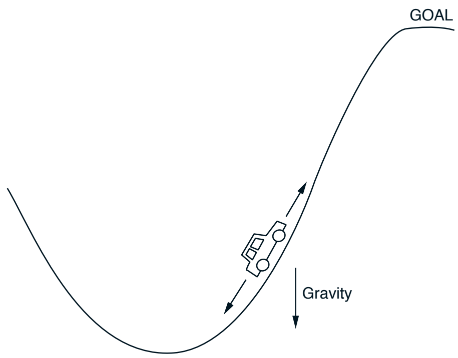
$$R_t = r_{t+1} + r_{t+2} + \cdots + r_T$$

  where T is the time when a terminal state is reached
- *Continuing tasks* (tasks which may go on forever):

$$R_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots = \sum_{k=1}^{\infty} \gamma^{t+k-1} r_{t+k}$$

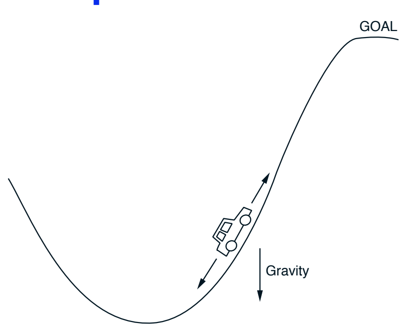
  Discount factor $\gamma < 1$ ensures that the return is finite, assuming that rewards are bounded.

Example: Mountain-Car



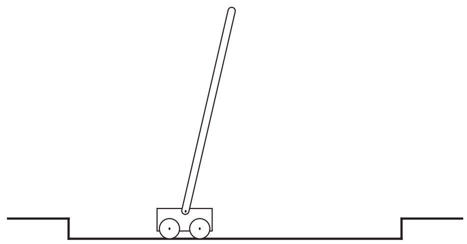
- States: position and velocity
- Actions: accelerate forward, accelerate backward, coast
- We want the car to get to the top of the hill as quickly as possible
- How do we define the rewards? What is the return?

Example: Mountain-Car



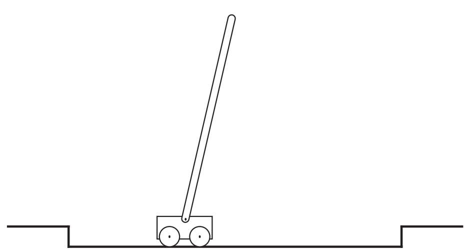
- States: position and velocity
- Actions: accelerate forward, accelerate backward, coast
- Two reward formulations:
  1. reward = -1 for every time step, until car reaches the top
  2. reward = 1 at the top, 0 otherwise $\gamma < 1$
- In both cases, the return is maximized by minimizing the number of steps to the top of the hill

Example: Pole Balancing



- We can push the cart along the track
- The goal is to avoid failure: pole falling beyond a given angle, or cart hitting the end of the track
- What are the states, actions, rewards and return?

Example: Pole Balancing



- States are described by 4 variables: angle and angular velocity of the pole relative to the cart, position and speed of cart along the track
- We can think of 3 possible actions: push left, push right, do nothing
- Episodic task formulation: reward = +1 for each step before failure
  $\Rightarrow$ return = number of steps before failure
- Continuing task formulation: reward = -1 upon failure, 0 otherwise, $\gamma < 1$
  $\Rightarrow$ return = $-\gamma^k$ if there are k steps before failure

Formulating Problems as MDPs

- The *rewards are quite "objective"* (unlike, e.g., heuristics), they are intended to capture the goal for the problem
- Often there are several ways to formulate a sequential decision problem as an MDP
- It is important that the state is defined in such a way that the Markov property holds
- Sometimes we may start with a more informative or lenient reward structure in the beginning, then change it to reflect the real task
- In psychology/animal learning, this is called *shaping*

Formulating Games as MDPs

- Suppose you played a game against a fixed opponent (possibly stochastic), which acts only based on the current board
- We can formulate this problem as an MDP by *making the opponent part of the environment*
- The states are all possible board positions for your player
- The actions are the legal moves in each state where it is your player's turn
- If we do not care about the length of the game, then $\gamma = 1$
- Rewards can be $+1$ for winning, -1 for losing, 0 for a tie (and 0 throughout the game)
- But it would be hard to define the transition probabilities!
- Later we will talk about how to learn such information from data/experimentation

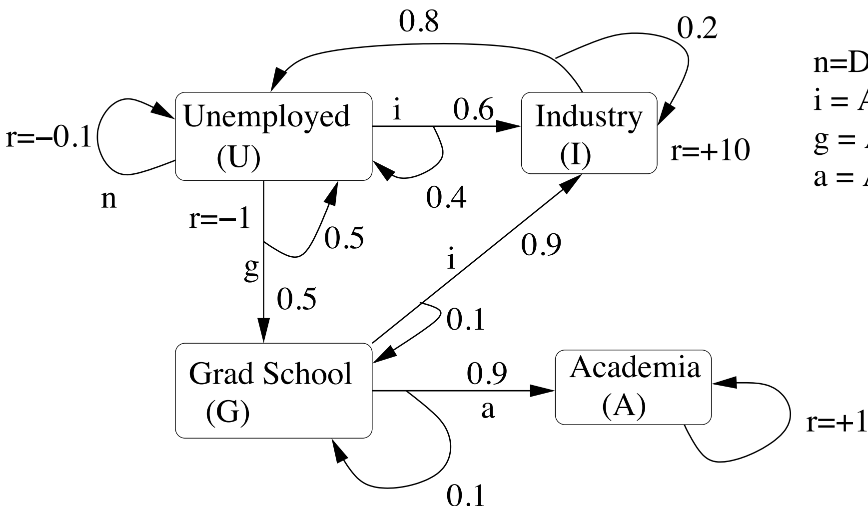
Policies

- The goal of the agent is to find a way of behaving, called a *policy* (plan or strategy) that maximizes the expected value of the return, $E[R_t], \forall t$
- A *policy* is a way of choosing actions based on the state:
  - *Stochastic policy*: in a given state, the agent can "roll a die" and choose different actions

$$\pi : S \times A \rightarrow [0, 1], \ \ \pi(s, a) = P(a_t = a | s_t = s)$$

  - *Deterministic policy*: in each state the agent chooses a unique action
    $\pi : S \rightarrow A, \ \ \pi(s) = a$

Example: Career Options



n=Do Nothing
i = Apply to industry
g = Apply to grad school
a = Apply to academia

What is the best policy?

Value Functions

- Because we want to find a policy which maximizes the expected return, it is a good idea to *estimate the expected return*
- Then we can *search* through the space of policies for a good policy
- *Value functions* represent the expected return, for every state, given a certain policy
- Computing value functions is an intermediate step towards computing good policies

State Value Function

- The *state value function of a policy* π is a function $V^\pi : S \to \mathbb{R}$
- The *value of state* s *under policy* π is the expected return if the agent starts from state s and picks actions according to policy π:

$$V^\pi(s) = E_\pi[R_t | s_t = s]$$

- For a finite state space, we can represent this as an array, with one entry for every state
- We will talk later about methods used for very large or continuous state spaces

Computing the value of policy π

- First, re-write the return a bit:

$$\begin{aligned} R_t &= r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots \\ &= r_{t+1} + \gamma \left(r_{t+2} + \gamma r_{t+3} + \cdots \right) \\ &= r_{t+1} + \gamma R_{t+1} \end{aligned}$$

- Based on this observation, V^π becomes:

$$V^\pi(s) = E_\pi[R_t | s_t = s] = E_\pi[r_{t+1} + \gamma R_{t+1} | s_t = s]$$

- Now we need to recall some properties of expectations...

Detour: Properties of expectations

- Expectation is *additive*: $E[X + Y] = E[X] + E[Y]$
  Proof: Suppose X and Y are discrete, taking values in $\mathcal{X}$ and $\mathcal{Y}$

$$\begin{aligned} E[X+Y] &= \sum_{x_i \in \mathcal{X}, y_i \in \mathcal{Y}} (x_i + y_i) p(x_i, y_i) \\ &= \sum_{x_i \in \mathcal{X}} x_i \sum_{y_i \in \mathcal{Y}} p(x_i, y_i) + \sum_{y_i \in \mathcal{Y}} y_i \sum_{x_i \in \mathcal{X}} p(x_i, y_i) \\ &= \sum_{x_i \in \mathcal{X}} x_i p(x_i) + \sum_{y_i \in \mathcal{Y}} y_i p(y_i) = E[X] + E[Y] \end{aligned}$$

- $E[aX] = aE[X]$ is $a \in \mathbb{R}$ is a constant
  Proof: $E[aX] = \sum_{x_i} a x_i p(x_i) = a \sum_{x_i} x_i p(x_i) = aE[X]$

Detour: Properties of expectations (2)

- The expectation of the product of random variables *is not* equal to the product of expectations, unless the variables are independent

$$E[XY] = \sum_{x_i \in \mathcal{X}, y_i \in \mathcal{Y}} x_i y_i p(x_i, y_i) = \sum_{x_i \in \mathcal{X}, y_i \in \mathcal{Y}} x_i y_i p(x_i | y_i) p(y_i)$$

- If X and Y are independent, then $p(x_i|y_i) = p(x_i)$, we can re-arrange the sums and products and get $E[X]E[Y]$ on the right-hand side
- But is X and Y are not independent, the right-hand side does not decompose!

Going back to value functions...

- We can re-write the value function as:

$$\begin{aligned} V^\pi(s) &= E_\pi[R_t | s_t = s] = E_\pi[r_{t+1} + \gamma R_{t+1} | s_t = s] \\ &= E_\pi[r_{t+1}] + \gamma E[R_{t+1} | s_t = s] \text{ (by linearity of expectation)} \\ &= \sum_{a \in A} \pi(s,a) r(s,a) + \gamma E[R_{t+1} | s_t = s] \text{ (by using definitions)} \end{aligned}$$

- The second term looks a lot like a value function, if we were to condition on s_{t+1} instead of s_t
- So we re-write as:

$$E[R_{t+1} | s_t = s] = \sum_{a \in A} \pi(s,a) \sum_{s' \in S} p(s,a,s') E[R_{t+1} | s_{t+1} = s']$$

- The last term is just $V^\pi(s')$

Bellman equations for policy evaluation

- By putting all the previous pieces together, we get:

$$V^\pi(s) = \sum_{a \in A} \pi(s,a) \left(r(s,a) + \gamma \sum_{s' \in S} p(s,a,s') V^\pi(s') \right)$$

- This is a *system of linear equations* (one for every state) whose unique solution is V^π.
- The uniqueness is ensured under mild technical conditions on the transitions p
- So if we want to find V^π, we could try to solve this system!

Iterative Policy Evaluation

- Main idea: turn Bellman equations into update rules.
  1. Start with some initial guess V_0
  2. During every iteration k, update the value function for all states:

$$V_{k+1}(s) \leftarrow \sum_{a \in A} \pi(s,a) \left(r(s,a) + \gamma \sum_{s' \in S} p(s,a,s') V_k(s') \right), \forall s$$

  3. Stop when the maximum change between two iterations is smaller than a desired threshold (the values stop changing)
- This is a *bootstrapping* algorithm: the value of one state is updated based on the current estimates of the values of successor states
- This is a dynamic programming algorithm
- If you have a linear system that is very big, using this approach avoids a big matrix inversion

Convergence of Iterative Policy Evaluation

- Consider the absolute error in our estimate $V_{k+1}(s)$:

$$
\begin{aligned}
|V_{k+1}(s) - V^\pi(s)| &= \Big| \sum_a \pi(s,a)(r(s,a) + \gamma \sum_{s'} p(s,a,s')V_k(s')) \\
&\quad - \sum_a \pi(s,a)(r(s,a) + \gamma \sum_{s'} p(s,a,s')V^\pi(s')) \Big| \\
&= \gamma \Big| \sum_a \pi(s,a) \sum_{s'} p(s,a,s')(V_k(s') - V^\pi(s')) \Big| \\
&\leq \gamma \sum_a \pi(s,a) \sum_{s'} p(s,a,s') |V_k(s') - V^\pi(s')|
\end{aligned}
$$

Convergence of Iterative Policy Evaluation (2)

- Let ϵ_k be the worst error at iteration k:

$$\epsilon_k = \max_{s' \in S} |V_k(s') - V^\pi(s')|$$

- From previous calculation, we have:

$$
\begin{aligned}
|V_{k+1}(s) - V^\pi(s)| &\leq \gamma \sum_a \pi(s,a) \sum_{s'} p(s,a,s') |V_k(s') - V^\pi(s')| \\
&\leq \gamma \sum_a \pi(s,a) \sum_{s'} p(s,a,s') \epsilon_k \\
&= \gamma \epsilon_k \sum_a \pi(s,a) \sum_{s'} p(s,a,s') \\
&= \gamma \epsilon_k \sum_a \pi(s,a) \cdot 1 = \gamma \epsilon_k, \forall s \in S
\end{aligned}
$$

Convergence of Iterative Policy Evaluation (3)

- Let $\epsilon_{k+1} = \max_s |V_{k+1}(s) - V^\pi(s)|$
- Since the previous inequality holds for all states, we have:

$$\epsilon_{k+1} \leq \gamma \epsilon_k$$

- Because $\gamma < 1$, this means that $\lim_{k \to \infty} \epsilon_k = 0$
- So, in the limit, we get the correct values
- More importantly, the error decreases exponentially
- We say that the error *contracts* and the contraction factor is γ.

Searching for a Good Policy

- We say that $\pi \geq \pi'$ if $V^\pi(s) \geq V^{\pi'}(s) \forall s \in S$
- This gives a partial ordering of policies: if one policy is better at one state but worse at another state, the two policies are incomparable
- Since we know how to compute values for policies, we can search through the space of policies
- Local search seems like a good fit.

Policy Improvement

$$V^\pi(s) = \sum_{a\in A} \pi(s,a) \left(r(s,a) + \gamma \sum_{s'\in S} p(s,a,s') V^\pi(s') \right)$$

- Suppose that there is some action a^*, such that:

$$r(s,a^*) + \gamma \sum_{s'\in S} p(s,a^*,s') V^\pi(s') > V^\pi(s)$$

- Then, if we set $\pi(s,a^*) \leftarrow 1$, the value of state s will increase
- This is because we replaced each element in the sum that defines $V^\pi(s)$ with a bigger value
- The values of states that can transition to s increase as well
- The values of all other states stay the same
- So the new policy using a^* is better than the initial policy π!

Policy iteration idea

- More generally, we can change the policy π to a new policy π', which is *greedy* with respect to the computed values V^π

$$\pi'(s) = \arg\max_{a\in A} \left(r(s,a) + \gamma \sum_{s'\in S} p(s,a,s') V^\pi(s') \right)$$

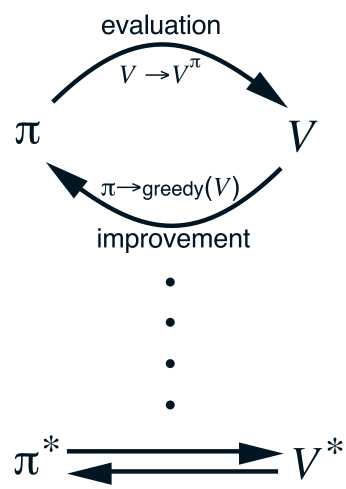
  Then $V^{\pi'}(s) \geq V^\pi(s), \forall s$
- This gives us a local search through the space of policies
- We stop when the values of two successive policies are identical

Policy Iteration Algorithm

1. Start with an initial policy π_0 (e.g., uniformly random)
2. Repeat:
   (a) Compute V^{π_i} using policy evaluation
   (b) Compute a new policy π_{i+1} that is greedy with respect to V^{π_i}

   until $V^{\pi_i} = V^{\pi_{i+1}}$

Generalized Policy Iteration



- In practice, we could run policy iteration incrementally
- Compute the value just to some approximation
- Make the policy greedy only at some states, not all states

Properties of policy iteration

- If the state and action sets are finite, there is a very large but finite number of deterministic policies
- Policy iteration is a greedy local search in this finite set
- We move to a new policy only if it provides a strict improvement
- So the algorithm *has to terminate*
- But if it is a greedy algorithm, can we guarantee an optimal solution?
- More on this next time...