Noname manuscript No.
(will be inserted by the editor)

On Dedekind's axiomatic approach to the foundations of mathematics

Dirk Schlimm

January 30, 2017

Abstract This paper brings together some points made previously in (Sieg and Schlimm 2005) and (Sieg and Schlimm 2014) focusing on some aspects of Dedekind's axiomatic approach to the foundations of mathematics. In particular the terminology for his axiomatic definitions of concepts and the development of the notion of mapping are discussed.

Keywords Axioms, Dedekind, mappings.

In recent years Dedekind's methodology and philosophical views have been construed in various ways. For example, he has been frequently described as being *conceptual* (Ferreirós 2007, 3) and *structuralist* (Reck 2003); a detailed study of the development of Dedekind's methodology and his meta-mathematic investigations of the foundations of arithmetic as being *axiomatic* was presented in (Sieg and Schlimm 2005), despite the fact that his texts do not follow the pattern of axioms–definitions–theorems that one might expect from an axiomatic exposition like that of Euclid or Hilbert (1899). While these construals might seem at first to stand in contrast with each other, they are in fact compatible, depending of course on a suitable understanding of what counts as being axiomatic. After all, axioms can be employed in more than one role in mathematical practice (Schlimm 2013). On the one hand, they typically serve as starting points for the derivation of theorems. On the other hand, they can also be used to define a class of models or, from a different perspective, as characteristic conditions [*Merkmale*] of a concept (whose instances satisfy the axioms). Through this second role of axioms, Dedekind's conceptual and structuralist methodology can be reconciled with an axiomatic approach, so that these do not exclude each other, but are seen as different aspects of the same practice.[1]

I will focus first on Dedekind's own terminology, because it differs somewhat from that used in the previous paragraph. Dedekind himself would not have called

McGill University, Montreal, Canada. E-mail: dirk.schlimm@mcgill.ca

[1] See also Sieg and Schlimm (2014) and Sieg and Morris (2016).

his own approach ‘axiomatic’, because of a different understanding of axioms than the contemporary one. In fact, he used the term ‘axiom’ only very rarely. Let us look more closely at these instances. In *Stetigkeit und irrationale Zahlen* (SZ), he writes that ‘The assumption of this property of the line [i. e., that every cut of the geometric line is determined by a point] is nothing but an axiom by which we attribute to the line its continuity, by which we think continuity into the line’ (1872, 18). For Dedekind this assumption is the essence of continuity and in the Preface he remarks that its content agrees with the ‘axiom’ given by Cantor (Dedekind 1872, 11). Cantor himself used this term, explaining that ‘I call this proposition an *axiom*, because it lies in its nature to be generally unprovable’ (Cantor 1872, 128, emphasis in original).[2] Here an axiom is not simply an assumption postulated as a starting point of the development a theory and which could be proved if other assumptions were chosen instead, but a proposition that is *essentially* unprovable. This could be, for example, because it is not a purely mathematical statement in the first place. Indeed, the only other occasion of Dedekind’s use of the term ‘*Axiom*’ is in a letter to his sister Mathilde from June 11, 1852. I quote the passage here in full, because it presents a personal side of Dedekind that is not seen very frequently:[3] ‘But how can I get into philosophy like that! What would Fichte say if he heard that an individual would be set before the others! After all, the fundamental principle of his philosophy is: “The I posits itself.” Think about how sharp-witted the man must have been who deduced our entire world order from this axiom. Yes, when I return I will philosophize to you something that will make your hair stand on end; unfortunately, however, I don’t understand much about it, which is supposed to be deplorable — but I don’t take it to heart too much, because I can be quite jolly without philosophy at all, as long as I receive my cello strings as soon as possible’ (Scharlau 1981, 33).[4] One might wonder where Dedekind’s interest in the idealist philosopher Fichte derives from and why it is mentioned in this passage at all. The answer to these questions can be found in Dedekind’s *Nachlass*, which contains notes from a lecture course on German philosophy held by Hermann Lotze that Dedekind attended in 1852 in Göttingen (Schlimm and Rudolph 2011). The proposition ‘The I posits itself’ [*Das Ich setzt sich selbst*], described as expressing Fichte’s condition for possibility and intelligibility of consciousness, was presented in a lecture held on June 10, 1852, just one day before Dedekind wrote the letter to his sister.

The above interpretation of Dedekind’s understanding of ‘axiom’ is also in accord with the terminology he used in other contexts. For example, when he introduces

[2] Dedekind notes that he received Cantor’s article on March 20, 1872, while writing the Preface to SZ. This does not settle definitively whether Dedekind’s use of ‘axiom’ on p. 18 was independent of Cantor’s or not. Cantor credits both Dedekind and himself for demanding the inclusion of ‘a certain axiom’ in the concept of a straight line in his letter to Dedekind from June 17, 1873 (Dugac 1976, 224).

[3] It is to the credit of the organizers of the conference *In Memoriam: Richard Dedekind (1831–1916)*, held in Braunschweig on October 6–8, 2016, to have arranged a program that highlighted many different aspects of the work and personal life of Dedekind.

[4] The original German is: Doch wie kann ich so in die Philosophie gerathen! Was würde Fichte sagen, wenn er hörte, dass ein Individuum vor die anderen gesetzt würde! Das Grundprinzip seiner Philosophie ist ja: “Das Ich setzt sich selbst”. Denke Dir, wie scharfsinnig der Mann gewesen sein muss, der aus diesem Axiome unsere ganze Weltordnung herleitet. Ja, wenn ich wiederkomme, werde ich Euch was vorphilosophieren, dass Euch die Haare zu Berge stehen; nur verstehe ich leider nicht recht viel davon, was ein Übelstand sein soll, den ich mir aber nicht sehr zu Herzen nehme, da ich auch ohne Philosophie ganz vergnügt sein kann, wenn ich citissime meine Cellosaiten erhalte.

the abstract concept of a lattice (called by Dedekind '*Dualgruppe*'), he gives what we would call an axiomatic, or structural, definition like the one now familiar for the definition of abstract groups (van der Waerden 1930, 15). Dedekind writes: 'A system $\mathfrak{A}$ of things $\alpha, \beta, \gamma \ldots$ is called a *Dualgruppe*, if there are two operations $\pm$, such that they create from two things α, β two things $\alpha \pm \beta$ that are also in $\mathfrak{A}$ and that satisfy the conditions [*Bedingungen*] A' (Dedekind 1897, 113). The 'conditions A' that express commutativity, associativity, and the absorption laws for the operations $-$ and $+$ were formulated earlier as 'fundamental laws [*Fundamentalgesetze*] A' (Dedekind 1897, 109), after the investigation of systems of numbers with the operations of greatest common divisor and least common multiple and of systems of elements with the operations of intersection and union.[5] Dedekind's introduction of simply infinite systems in # 71 of *Was sind und was sollen die Zahlen?* (WZ) (1888, 359), where a system N has to satisfy certain conditions [*Bedingungen*] $\alpha, \beta, \gamma, \delta$, is along the same lines. Finally, the general applicability of the four arithmetical operations is called the 'fundamental property' of the notion of a number field (Dirichlet 1879, 435) and Dedekind later refers to it as a 'characteristic condition' [*Merkmal*] of a number field (Dedekind 1872, 318). We see from these examples that if axioms are used for the definitions of concepts, i. e., in a semantic role, Dedekind typically refers to them as laws, conditions, or characteristic conditions. This way of proceeding bears great affinity to Hilbert's axiomatic approach, as can easily be seen by comparing Dedekind's structural definitions with the opening paragraphs of Hilbert's *Grundlagen der Geometrie* (1899), with the difference that Hilbert calls the conditions that a system of geometry has to satisfy 'axioms'. He explains this choice of terminology in a letter to Frege (December 29, 1899) by referring to the customary usage by mathematicians and physicists, but also makes clear that he considers this a purely terminological issue: 'The renaming of "axioms" as "characteristic conditions" [*Merkmale*] etc. is a pure formality and, in addition, a matter of taste — in any event, it is easily accomplished' (Frege 1976, 66).

Dedekind is fully aware of the dual roles of axioms mentioned above, namely that they can be definitions of concepts and starting points for derivations. For example, for the notion of continuity Dedekind seeks 'a precise characteristic condition [*Merkmal*] [. . .] that can be used as the basis for actual deductions' (Dedekind 1872, 322). It is worth noting that in some cases Dedekind also uses a particular term for axioms that are used mainly in their syntactic role, i. e., as starting points for deductions. A short manuscript on projective geometry, which has been preserved in Dedekind's *Nachlass*, is titled 'The assumptions [*Voraussetzungen*] of pure geometry of position and its relations to the science of numbers' (Dedekind unk).[6] Here Dedekind is not so much interested in defining a projective space, but in deriving theorems. From a modern perspective, Dedekind's assumptions, then, are axioms like Euclid's; accordingly, Scharlau, who produced an inventory of Dedekind's *Nachlass*, describes the content of the manuscript as 'An attempt at an axiomatic geometry' (*Versuch einer axiomatischen Geometrie*) (Scharlau 2010, 64).

Let us now turn briefly to the question of why Dedekind did not prove the categoricity of the axiomatic characterization of the real numbers in SZ (1872), although he did this for the natural numbers in 1888. Central to the meta-mathematical in-

[5] See also (Schlimm 2011, 53–57).

[6] I agree with Ferreirós' dating of this manuscript as being written in the late 1870s or 1880s, based on the terminology used by Dedekind (Ferreirós 2007, 238); possibly based on its subject matter, Scharlau dates it as originating in the 1860s.

vestigations that underlie his mature mathematical and philosophical outlook is the notion of mapping [*Abbildung*], which is presented in Dedekind's WZ (1888). However, this notion did not suddenly appear fully-formed in 1888, but is the result of a continuous development that can be traced back to the earliest writings of Dedekind, namely his *Habilitationsrede* (1854) and the lecture notes on group theory and algebra (1855–58).[7] A careful discussion of this development, with particular attention to Dedekind's work on the real numbers, algebraic number theory, various drafts for WZ, and the correspondence with Cantor, can be found in (Sieg and Schlimm 2014). What follows is a short summary. To trace this development and to distinguish the different conceptions of mappings that can be identified in Dedekind's writings, it is useful to consider the kinds of elements that can be used as domain and range of functions and mappings. Moving away from considering only numbers as possible domains and ranges for functions, Dedekind mentions possible correspondences between different kinds of objects in 1872, but the first time that Dedekind speaks of a 'mapping' between different kinds of systems is not until 1888. One can also detect a development in how Dedekind treats functions and mappings as genuine objects of investigation. Despite using homomorphisms implicitly in his early algebraic notes, it was only in 1877, i. e., when SZ (1872) was already written, but before the publication of the axiomatic presentation of the natural numbers in 1888, that Dedekind discussed for the first time in print the properties of mappings and formulated explicitly the properties of injectivity and surjectivity. Dedekind's draft of WZ from 1872–78 provides the first evidence for the development of the conceptual apparatus that is needed in order to prove that two models of an axiom system that belong to different domains of objects have the same structure (i. e., that they are isomorphic). Thus, it was only after 1872 that Dedekind arrived at a rigorous and general concept of mapping that allows for different kinds of objects to be mapped to each other. This point is crucial for extending the interpretation of Dedekind's methodology as being axiomatic also to his earlier work on the real numbers and allows us to explain the lack of a categoricity result for the real numbers in Dedekind's 1872 publication.

To conclude, we have seen above that Dedekind gave structural definitions of abstract concepts like that of a simply infinite system, but that he did not use the term '*Axiom*' for the characteristic conditions. He also considered systems that instantiate abstract concepts and that satisfy the corresponding axioms (nowadays called 'models'), and by 1888 had developed a rigorous notion of isomorphisms between models in terms of mappings. Thus, Dedekind developed all the ingredients for an axiomatic approach and used them skillfully for the definition of abstract concepts and their metatheoretic study.

Acknowledgements. The author would like to thank Daniel Lovsted for comments on an earlier draft of this paper.

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[7] See Dedekind (1855) and Scharlau (1981, 59–108).

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