SORTING + STRING

COMP 321 – McGill University

These slides are mainly compiled from the following resources.

- Professor Jaehyun Park' slides CS 97SI
- Top-coder tutorials.
- Programming Challenges book.

SORTING

- Practical applications in computing require things to be in order.
- To consider:
  - Runtime.
  - Memory Space.
    - In-place algorithms ???
  - Stability.
    - What happens to elements that are comparatively the same?

SORTING

- Practical applications in computing require things to be in order.
- To consider:
  - Runtime.
  - Memory Space.
    - In-place algorithms => without creating copies of the data
  - Stability.
    - What happens to elements that are comparatively the same?
    - Those elements whose comparison key is the same will remain in the same relative order after sorting as they were before sorting.

Bubble Sort

- To pass through the data and swap two adjacent elements whenever the first is greater than the last. Thus, the smallest elements will “bubble” to the surface.
- $O(n^2)$.
- Simple to understand and code from memory + Stable + In-place.

```
for (int i = 0; i < data.Length; i++)
   for (int j = 0; j < data.Length - 1; j++)
      if (data[j] > data[j + 1])
      {
         tmp = data[j];
         data[j] = data[j + 1];
         data[j + 1] = tmp;
      }
```

Insertion Sort

- It seeks to sort a list one element at a time. With each iteration, it takes the next element waiting to be sorted, and adds it, in proper location, to those elements that have already been sorted.
- $O(n^2)$.
- it works very efficiently for lists that are nearly sorted initially

```
for (int i = 0; i <= data.Length; i++) {
    int j = i;
    while (j > 0 && data[i] < data[j - 1])
        j--;
    int tmp = data[i];
    for (int k = i; k > j; k--)
        data[k] = data[k - 1];
    data[j] = tmp;
}
```

Merge Sort

- A merge sort works recursively (divide and conquer). Divide the unsorted list into n sublists, each containing 1 element. Then, merge sublists to produce new sorted sublists.
- O(n log n).
- Faily efficient + can be used to solve other problems.

```
int[] mergeSort (int[] data) {
   if (data.Length == 1)
      return data;
   int middle = data.Length / 2;
   int[] left = mergeSort(subArray(data, 0, middle - 1));
   int[] right = mergeSort(subArray(data, middle, data.Length - 1));
   int[] result = new int[data.Length];
```

Heap Sort

- All data from a list is inserted into a heap, and then the root element is repeatedly removed and stored back into the list.
- O(nlogn)
- Not stable

```
Heap h = new Heap();
for (int i = 0; i < data.Length; i++)
    h.Add(data[i]);
int[] result = new int[data.Length];
for (int i = 0; i < data.Length; i++)
    data[i] = h.RemoveLowest();
```

Quick Sort

- Divide the data into two groups of “high” values and “low” values. Then, recursively process the two halves. Finally, reassemble the now sorted list.
- $O(n^2)$
- dependent upon how successfully an accurate midpoint value is selected

```
Array quickSort(Array data) {
   if (Array.Length <= 1)
      return;
   middle = Array[Array.Length / 2];
   Array left = new Array();
   Array right = new Array();
   for (int i = 0; i < Array.Length; i++)
      if (i != Array.Length / 2) {
         if (Array[i] <= middle)
            left.Add(Array[i]);
         else
            right.Add(Array[i]);
      }
   return concatenate(quickSort(left), middle, quickSort(right));
}
```

Radix Sort

- Sort data without having to directly compare elements to each other. It groups keys by the individual digits which share the same significant position and value.
- O(n * k), where k is the size of the key.
- Some types of data may use very long keys, or may not easily lend itself to a representation that can be processed

Sorting Libraries

- Java API, and C++ STL all provide some built-in sorting capabilities.
- Check the interface called Comparable => you add a method int CompareTo (object other), which returns a negative value if less than, 0 if equal to, or a positive value if greater than the parameter.
- Also check the interface called Comparator. which defines a single method int Compare (object obj1, object obj2), which returns a value indicating the results of comparing the two parameters.

STRINGS

String Matching Problem

- Given a text T and a pattern P, find all occurrences of P within T
- Notations:
  - n and m : lengths of P and T
  - : set of alphabets (of constant size)
  - P_i : i th letter of P (1-indexed)
  - a , b , c : single letters in
  - x , y , z : strings

String Matching Problem

- T = AGCATGCTGCAGTCATGCTTAGGCTA
- P = GCT
- P appears three times in T
- A naive method takes O(mn) time
  - Initiate string comparison at every starting point
  - Each comparison takes O(m) time
- We can do much better!

String Matching Problem - Hash

- Main idea: preprocess T to speedup queries
  - Hash every substring of length k
  - k is a small constant
- For each query P, hash the first k letters of P to retrieve all the occurrences of it within T
- Don’t forget to check collisions!

String Matching Problem - Hash

- Pros:
  - Easy to implement
  - Significant speedup in practice
- Cons:
  - Doesn’t help the asymptotic efficiency
    - Can still take O(nm) time if hashing is terrible or data is difficult
    - Can you give me an example of the worst case?
- A lot of memory consumption

String Matching Problem - Hash

- Pros:
  - Easy to implement
  - Significant speedup in practice
- Cons:
  - Doesn’t help the asymptotic efficiency
    - Can still take O(nm) time if hashing is terrible or data is difficult
    - Can you give me an example of the worst case? => When all the characters of pattern and text are same. T=AAAAAAA… P=AAA.
- A lot of memory consumption

SMP - Knuth-Morris-Pratt (KMP)

- A linear time (!) algorithm that solves the string matching problem by preprocessing P in O(m) time
  - Main idea is to skip some comparisons by using the previous comparison result.
- Uses an auxiliary array π that is defined as the following:
  - π[i] is the largest integer smaller than i such that $P_1 \ldots P_{\pi[i]}$ is a suffix of $P_1 \ldots P_i$
    - e.g., π[6] = 4 since abab is a suffix of ababab
    - e.g., π[9] = 0 since no prefix of length ≤ 8 ends with c

i	1	2	3	4	5	6	7	8	9	10
P_i	a	b	a	b	a	b	a	b	c	a
$\pi[i]$	0	0	1	2	3	4	5	6	0	1

Question for you

- Why is π useful?

SMP - Knuth-Morris-Pratt (KMP)

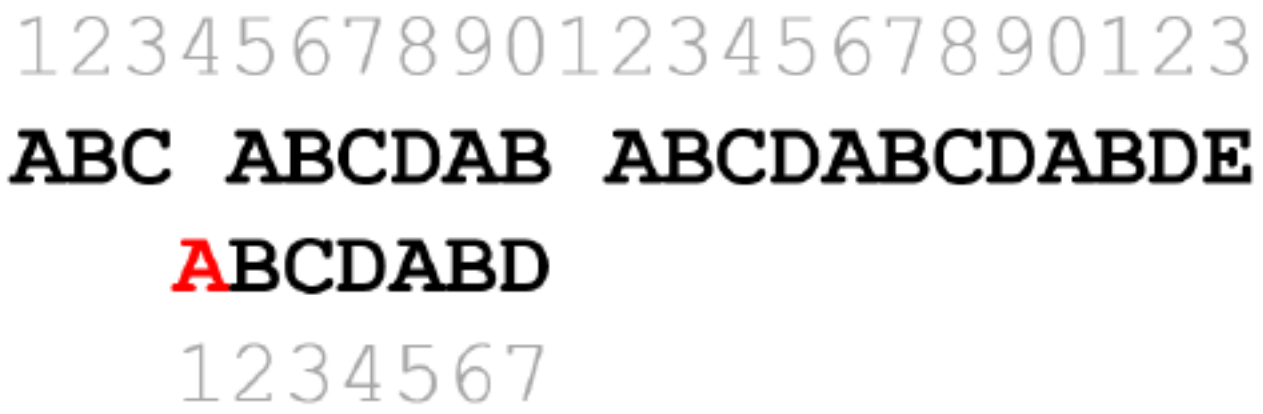
- T = ABC ABCDAB ABCDABCDABDE
- P = ABCDABD
- π = (0,0,0,0,1,2,0)
- Start matching at the first position of T:

```
12345678901234567890123
ABC ABCDAB ABCDABCDABDE
ABCDABD
1234567
```

- Mismatch at the 4th letter of P!

SMP - Knuth-Morris-Pratt (KMP)

- We matched k = 3 letters so far, and π[k] = 0
  - Thus, there is no point in starting the comparison at T2 , T3
- Shift P by k − π[k] = 3 letters



- Mismatch at the 4th letter of P!

SMP - Knuth-Morris-Pratt (KMP)

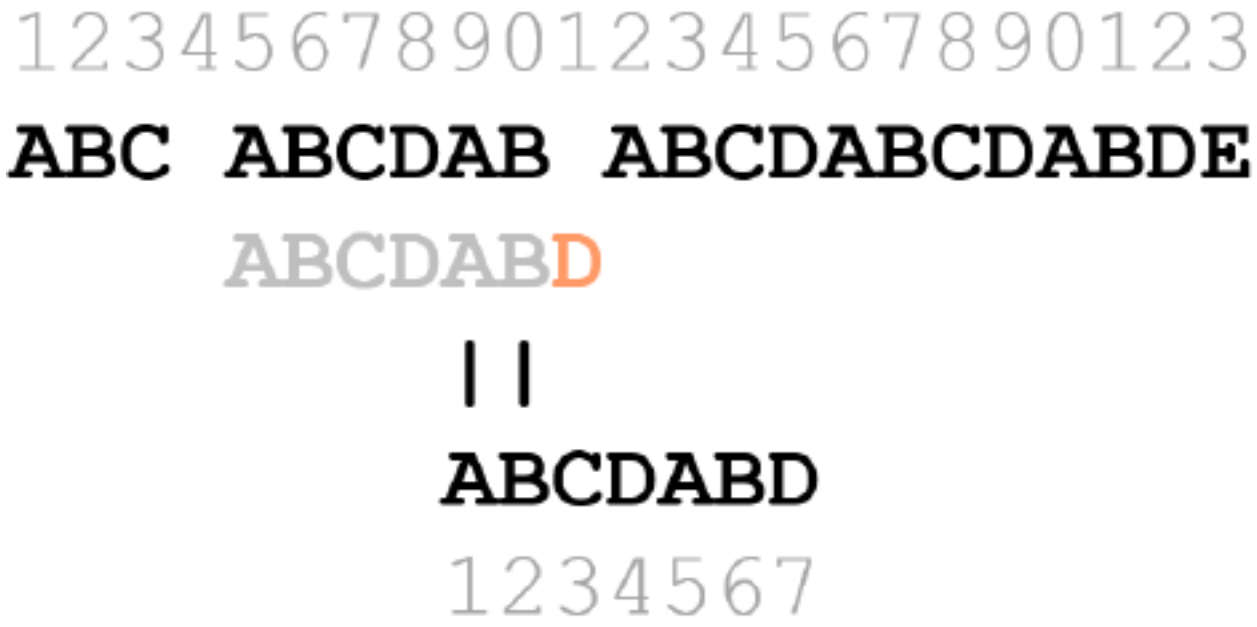
- We matched k = 0 letters so far
- Shift P by k−π[k] = 1 letter (we define π[0] = −1)

```
          12345678901234567890123
          ABC  ABCDAB  ABCDABCDABDE
               ABCDABD
               1234567
```

- Mismatch at I_{11}!

SMP - Knuth-Morris-Pratt (KMP)

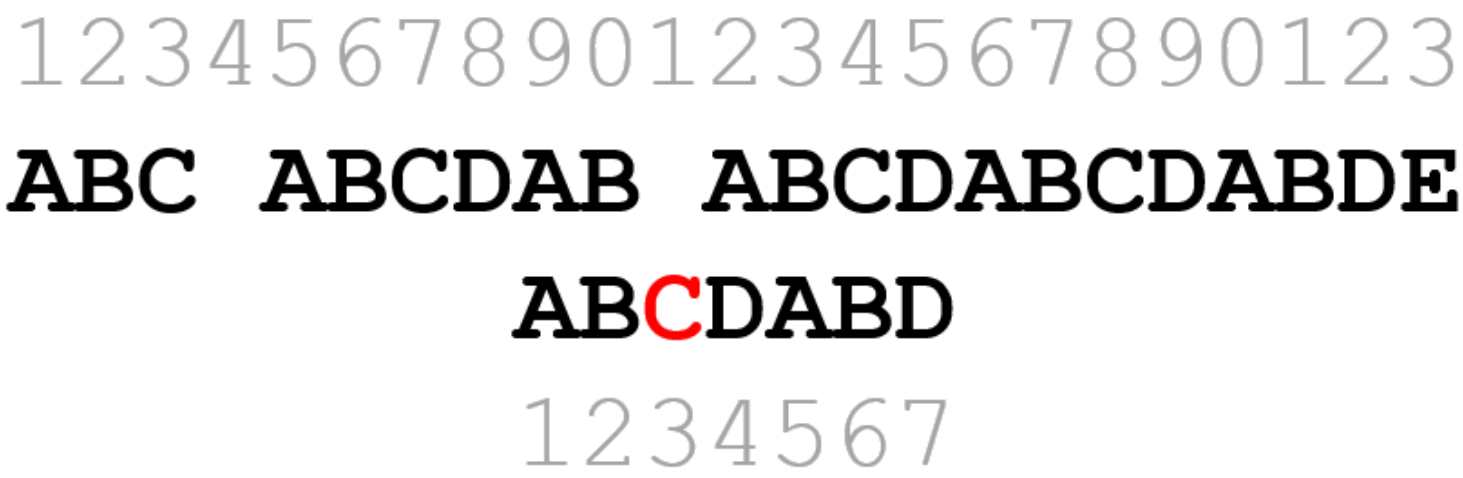
- $\pi[6] = 2$ means P_1P_2 is a suffix of $P_1 \ldots P_6$
- Shift P by $6 - \pi[6] = 4$ letters



- Again, no point in shifting P by 1, 2, or 3 letters

SMP - Knuth-Morris-Pratt (KMP)

- Mismatch at T_{11} again!



- Currently 2 letters are matched
- Shift P by 2 − π[2] = 2 letters

SMP - Knuth-Morris-Pratt (KMP)

- Mismatch at T_{11} again!

```
12345678901234567890123
ABC  ABCDAB  ABCDABCDABDE
             ABCDABD
             1234567
```

- Currently no letters are matched
- Shift P by 0 − π[0] = 1 letter

SMP - Knuth-Morris-Pratt (KMP)

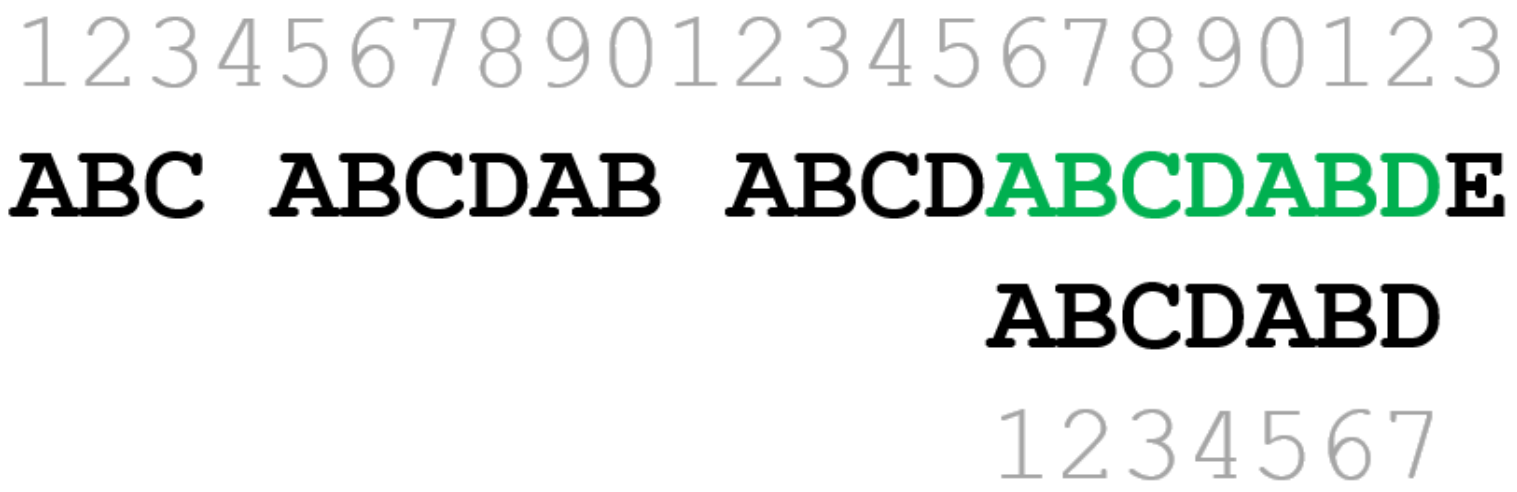
- Mismatch at T18

```
12345678901234567890123
ABC  ABCDAB  ABCDABCDABDE
             ABCDABD
             1234567
```

- Currently 6 letters are matched
- Shift P by 6 − π[6] = 4 letters

SMP - Knuth-Morris-Pratt (KMP)

- Finally, there it is!



- After recording this match (at T16 . . . T22, we shift P again in order to find other matches
  - Shift by 7 − π[7] = 7 letters

SMP - Knuth-Morris-Pratt (KMP)

- Computing π.
- Obs1=> if $P_1 \ldots P_{\pi[i]}$ is a suffix of $P_1 \ldots P_i$, then $P_1 \ldots P_{\pi[i]-1}$ is a suffix of $P_1 \ldots P_{i-1}$
- Obs2 => all the prefixes of P that are a suffix of $P_1 \ldots P_i$ can be obtained by recursively applying to I
  - e.g. , $P_1 \ldots P_{\pi[i]}$, $P_1 \ldots$, $P_{\pi[\pi[i]]}$, P1 . . . , , $P_{\pi[\pi[\pi[i]]]}$ are all suffixes of $P_1 \ldots P_i$

SMP - Knuth-Morris-Pratt (KMP)

- Computing π.
- Obs3 (not obvious) =>
  - First, let's write $\pi^{(k)}[i]$ as π[.] applied k times to I
    - e.g., $\pi^{(2)}[i] = \pi[\pi[i]]$
  - π[i] is equal to $\pi^{(k)}[i - 1] + 1$, where k is the smallest integer that satisfies $P_{\pi(k)[i-1]+1} = P_i$
    - If there is no such k, [i] = 0
- Intuition: we look at all the prefixes of P that are suffixes of $P_1 \ldots P_{i-1}$, and find the longest one whose next letter matches P_i

SMP - Knuth-Morris-Pratt (KMP)

- Implementation π.

```
pi[0] = -1;
int k = -1;
for(int i = 1; i <= m; i++) {
  while(k >= 0 && P[k+1] != P[i])
    k = pi[k];
  pi[i] = ++k;
}
```

SMP - Knuth-Morris-Pratt (KMP)

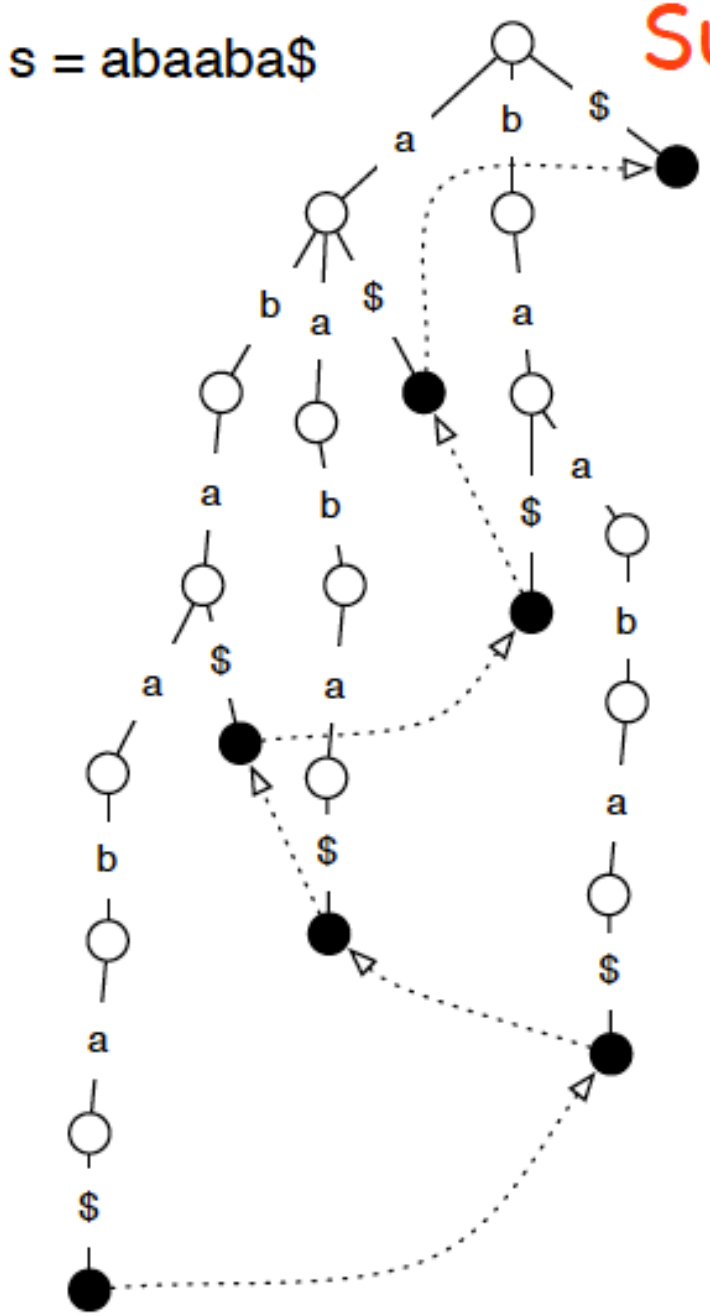
- Implementation KMP.

```
int k = 0;
for(int i = 1; i <= n; i++) {
  while(k >= 0 && P[k+1] != T[i])
    k = pi[k];
  k++;
  if(k == m) {
    // P matches T[i-m+1..i]
    k = pi[k];
  }
}
```

SMP – Suffix trie

- Suffix trie of a string T is a rooted tree that stores all the suffixes (thus all the substrings)
- Each node corresponds to some substring of T
- Each edge is associated with an alphabet
- For each node that corresponds to ax, there is a special pointer called suffix link that leads to the node corresponding to x
- Surprisingly easy to implement!

Suffix Links



- Suffix links connect node representing "xα" to a node representing "α".
- Most important suffix links are the ones connecting suffixes of the full string (shown at right).
- But every node has a suffix link.
  - Why?
  - How do we know a node representing α exists for every node representing xα?

Applications of Suffix Tries (1)

Check whether q is a **substring** of T:

Follow the path for q starting from the root.
If you exhaust the query string, then q is in T.

Check whether q is a **suffix** of T:

Follow the path for q starting from the root.
If you end at a leaf at the end of q, then q is a suffix of T

Count # of occurrences of q in T:

Follow the path for q starting from the root.
The number of leaves under the node you end up in is the number of occurrences of q.

Find the longest repeat in T:

Find the deepest node that has at least 2 leaves under it.

Find the lexicographically (alphabetically) first suffix:

Start at the root, and follow the edge labeled with the lexicographically (alphabetically) smallest letter.

Suppose we want to build suffix trie for string:

s = abbacabaa

We will walk down the string from left to right:

abbacabaa

building suffix tries for s[0], s[0..1], s[0..2], ..., s[0..n]

To build suffix trie for s[0..i], we will use the suffix trie for s[0..i-1] built in previous step

To convert SufTrie(S[0..i-1]) → SufTrie(s[0..i]), add character s[i] to all the suffixes:

abbacabaa
i=4

Need to add nodes for the suffixes:

abbac
bbac
bac
ac
c

Purple are suffixes that will exist in SufTrie(s[0..i-1]) Why?

How can we find these suffixes quickly?

abbacabaa

i=4

Need to add nodes for the suffixes:

abbac

bbac

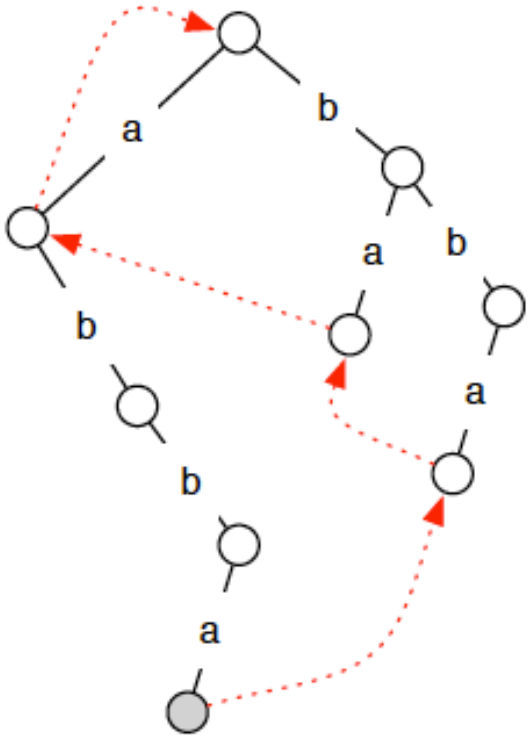
bac

ac

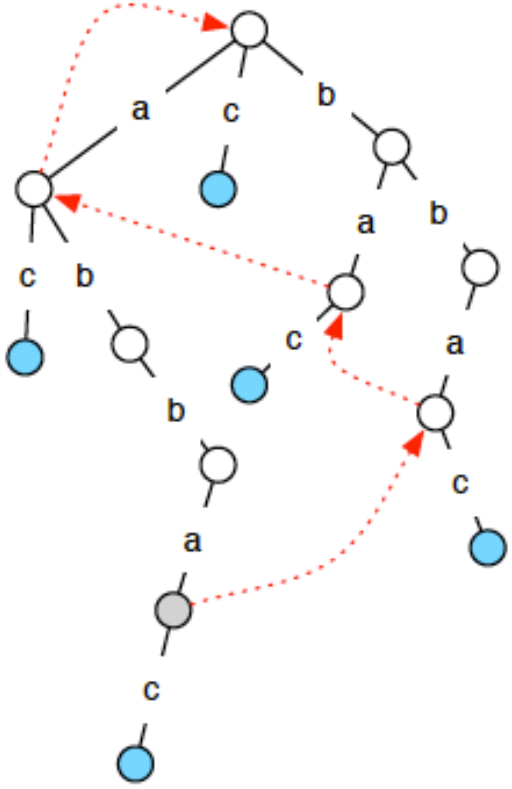
c

Purple are suffixes that will exist in SufTrie(s[0..i-1]) Why?

How can we find these suffixes quickly?



SufTrie(abba)



SufTrie(abbac)

Where is the new deepest node? (aka longest suffix)

How do we add the suffix links for the new nodes?

To build SufTrie(s[0..i]) from SufTrie(s[0..i-1]):

CurrentSuffix = longest (aka deepest suffix)

Repeat:
Add child labeled s[i] to CurrentSuffix.
Follow suffix link to set CurrentSuffix to next shortest suffix.

until you reach the root or the current node already has an edge labeled s[i] leaving it.

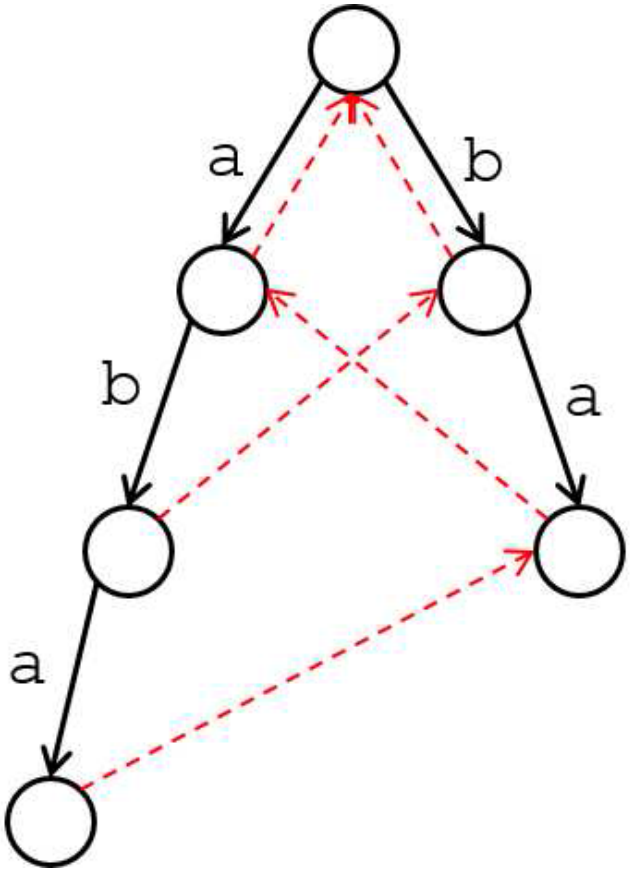
Add suffix links connecting nodes you just added in the order in which you added them.

Because if you already have a node for suffix αs[i] then you have a node for every smaller suffix.

In practice, you add these links as you go along, rather than at the end.

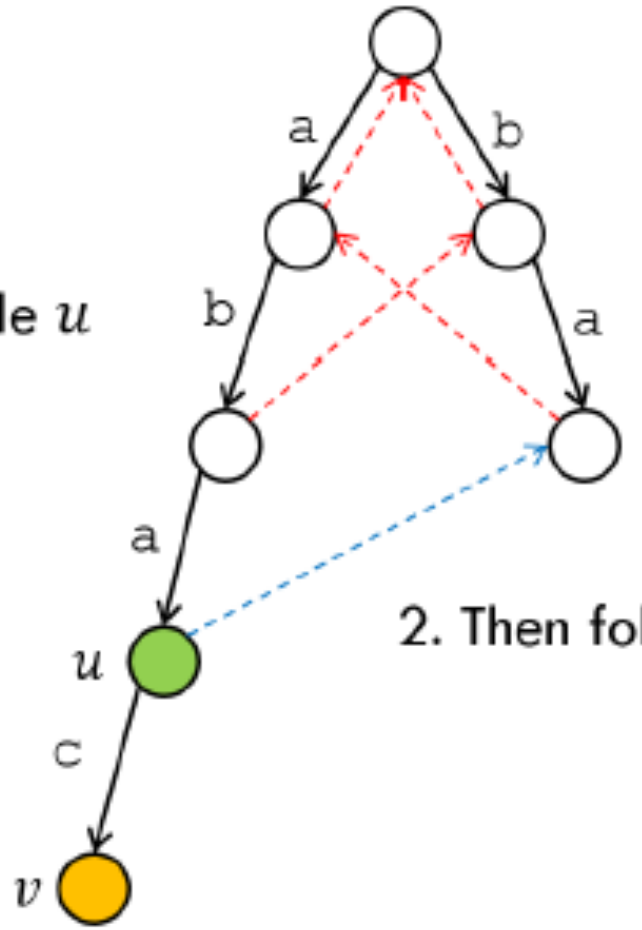
SMP – Suffix trie

- Given the suffix trie for aba, we want to add a new letter c



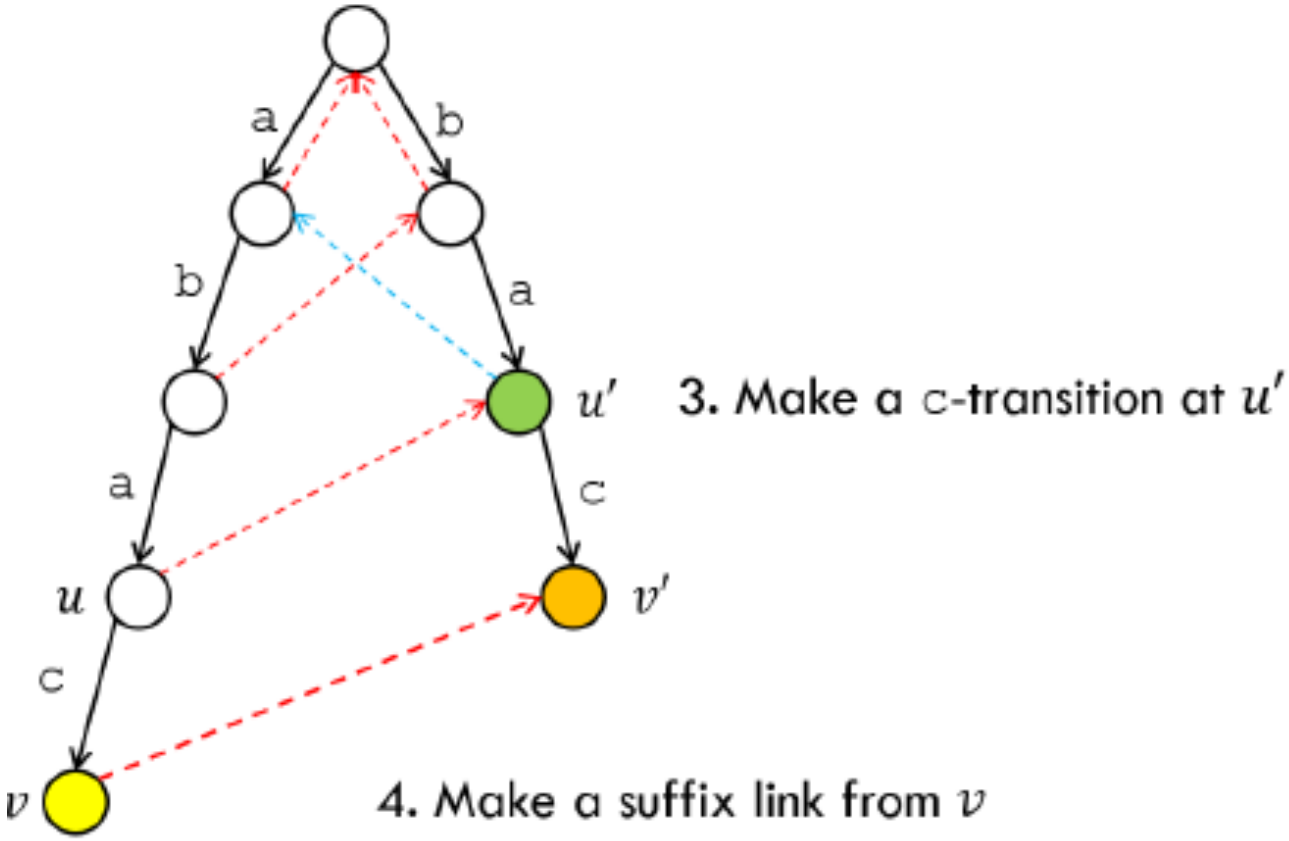
SMP – Suffix trie

1. Start at the green node u and make a c-transition

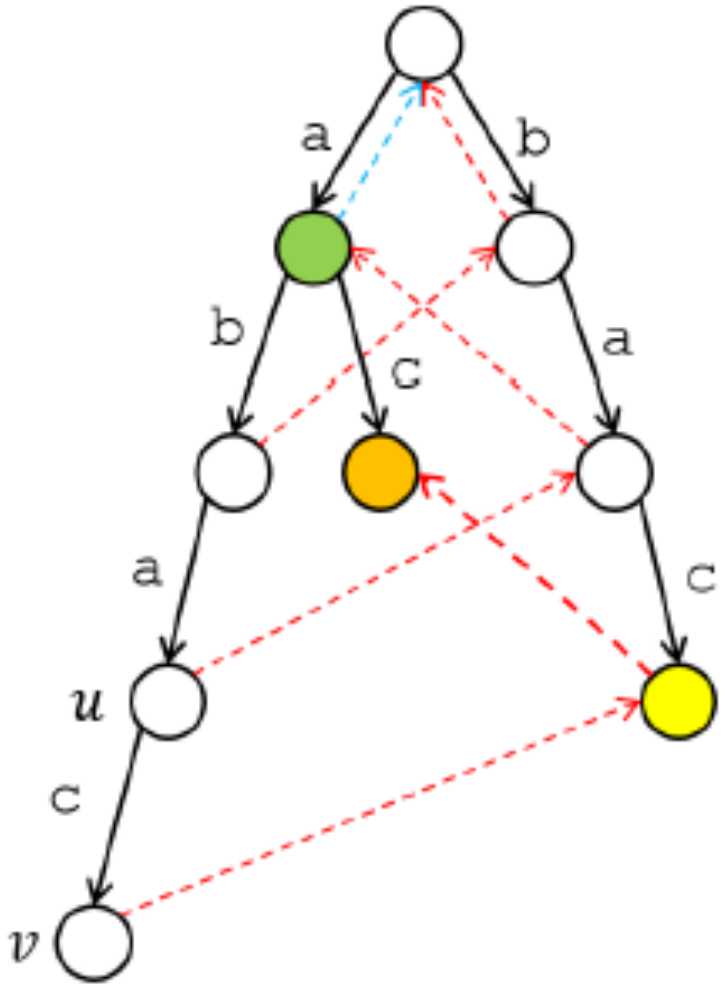


2. Then follow the suffix link

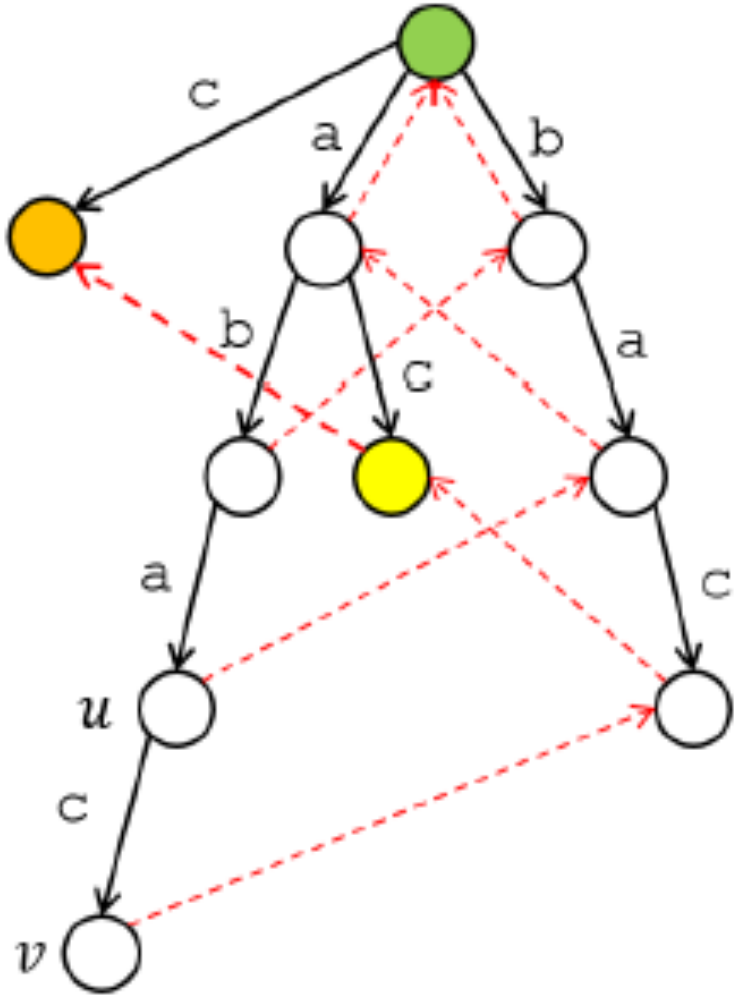
SMP – Suffix trie



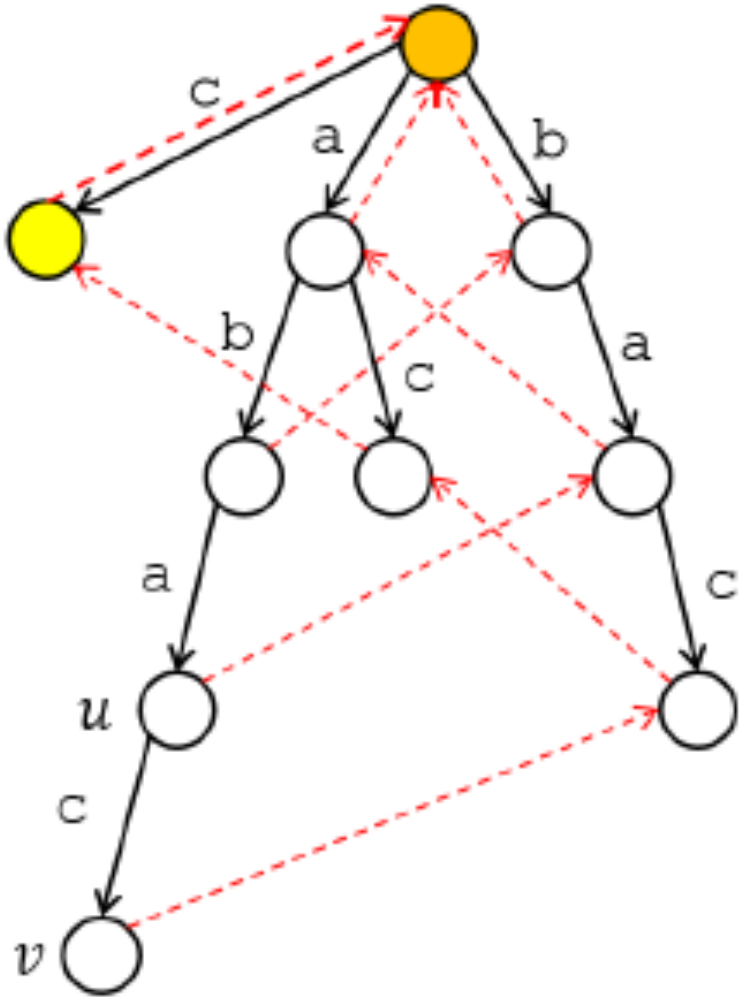
SMP – Suffix trie



SMP – Suffix trie



SMP – Suffix trie



SMP-Suffix Array

Input string	Get all suffixes	Sort the suffixes	Take the indices
BANANA	1 BANANA 2 ANANA 3 NANA 4 ANA 5 NA 6 A	6 A 4 ANA 2 ANANA 1 BANANA 5 NA 3 NANA	6,4,2,1,5,3

SMP – Suffix Array

- Memory usage is O(n)
- Has the same computational power as suffix trie
- Can be constructed in O(n) time (!)
  - But it's hard to implement

Notes

- Always be aware of the null-terminators
- Simple hash works so well in many problems
- If a problem involves rotations of some string, consider concatenating it with itself and see if it helps
- It is a smart idea to have the implementation of suffix arrays and KMP in your notebook.