Instructions(important!): You are allowed to work with your classmates on solving assignments, but you must write your own solutions independently.

If you read a solution from other resources (e.g. books, articles, ...), you should cite each one of the resources. In this case, try to understand the solution first, and then write down your solution without looking at the resources.

Present your solution in a clear and rigorous way. Clarity is as important as correctness. You will lose points for a correct solution that is poorly presented.

If you can not find a solution of a problem, do not answer it, this will earn you 20% of the points. If you have some non-trivial yet incomplete ideas to solve a problem, write them down and indicate the gaps.

Submit your solution in class or to yaqiao.li@mail.mcgill.ca. Late submission will not be accepted.

Problems (each worth 20 points):

1. (Hilbert’s 10th problem) Let \( p : \mathbb{R}^n \to \mathbb{R} \) be a polynomial with \( n \) variables, say \( x \in \mathbb{R}^n \) is an integral root of \( p \) if \( p(x) = 0 \) and \( x \in \mathbb{Z}^n \).

   (1) [6 points] Consider the language
   \[
   L_n := \{ p : \mathbb{R}^n \to \mathbb{R} \mid p \text{ is a polynomial with integer coefficients, and has an integral root} \}.
   \]
   Show that \( L_n \) is Turing-recongnizable.

   (2) [8 points] Let \( p : \mathbb{R} \to \mathbb{R} \) be a polynomial with one single variable defined as \( p(x) = \sum_{i=0}^{d} c_i x^i \). Let \( c_{\text{max}} := \max_{0 \leq i \leq d} |c_i| \). Show that any root \( y \) of \( p \) satisfies,
   \[
   |y| \leq (d + 1) \frac{c_{\text{max}}}{|c_d|}.
   \]

   (3) [6 points] Show that \( L_1 \) is Turing-decidable. Matiyasevich has shown that \( L_n \) is in general not decidable.

2. (Computational power) As usual, DFA denotes deterministic finite automaton, and TM denotes Turing machine.

   (1) [10 points] In terms of computational power, which one is stronger between DFA and a standard TM? Argue rigorously.

   (2) [10 points] We know the concept of TM is quite robust. Consider a TM with one tape, and with the restriction that its head can only moves to the right, is it equivalent to the standard TM? Argue rigorously.

3. (Space complexity) Suppose that \( f \) and \( g \) are computable in logarithmic space, show that \( f \circ g \) is computable in logarithmic space too.

4. (Polynomial hierarchy) Show the following:
(1) [10 points] If \( \mathbf{P} = \mathbf{NP} \), then \( \mathbf{P} = \mathbf{PH} \).

(2) [10 points] If \( \mathbf{PH} = \mathbf{PSPACE} \), then \( \mathbf{PH} = \sum_k^n \) for some \( k \).

5. (Communication complexity) Recall \( D(f) \) denotes the deterministic communication complexity of a function \( f \). The famous log rank conjecture says that \( D(f) \) and \( \log \text{rank}(f) \) are polynomially related for all boolean-valued functions \( f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\} \). The following problem is to show that the conjecture is NOT true if we allow the boolean function \( f \) to be real valued.

Let \( \text{INTS} : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1,2,\ldots,n\} \) be the size-intersection function defined as
\[
\text{INTS}(x,y) = \sum_{i=1}^n x_i y_i.
\]

Intuitively, \( \text{INTS}(x,y) \) is simply the size of the intersection of \( x \) and \( y \) if we think of \( x \) and \( y \) as subsets of \( \{1,2,\ldots,n\} \). Show

(1) [10 points] \( D(\text{INTS}) \geq n \).

(2) [10 points] \( \text{rank}(\text{INTS}) = n \).

This concludes that \( D(\text{INTS}) \geq 2^{\log \text{rank}(\text{INTS})} \).