1. The probabilistic method is a technique to prove the existence of objects with desired combinatorial properties. The idea is to define a suitable probability distribution in which the probability of finding the desired properties is nonzero. As an illustration of the method, consider the following exercise: Let $G = (V, E)$ be any graph that has a matching $M$ (a matching is a collection of edges, no two of which are adjacent to the same vertex). Show that $G$ contains a subgraph $H$, where $H$ is bipartite and contains at least $\frac{|E| + |M|}{2}$ edges.

(Hint: Think of a random bipartition scheme of the set of vertices of $G$ such that you guarantee that each edge of the matching has its endpoints on opposite edges of the partition. Calculate the expected number of edges of $G$ that have their endpoints in opposite partitions.)

2. Consider a problem $\{f_n\}$ such that we have a randomized protocol $P$ with

$$Pr[P(x, y, r) \neq f_n(x, y)] \leq \frac{1}{3}$$

Show that there is a randomized protocol $Q(x, y, r)$ which uses only $|r| = O(\log n)$ random bits, and still achieves

$$Pr[P(x, y, r) \neq f_n(x, y)] \leq \frac{1}{3}$$

Hint: Use the probabilistic method, along with Chernoff bounds, as in the proof that $R^{priv}(f) \leq R_e(f) + O(\log(n) + \log(\frac{1}{\delta}))$

3. Let $DISJ : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\}$ be defined by

$$DISJ(x, y) = \begin{cases} 1 & \text{if } x \cap y = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

In the above we view the inputs as subsets of $\{1, \ldots, n\}$. Show that $D(DISJ) = n + 1$.

4. Prove that if $NP \subseteq BPP$ then $NP = RP$.

5. The class PP is defined as the set of languages $L$ such that there exists a polynomial $q(n)$ and a polynomial time machine $M$ such that the following holds.

$$x \in L \implies Pr_{r \in \{0, 1\}^{q(n)}}[M(x, r) = 1] > \frac{1}{2}$$

$$x \notin L \implies Pr_{r \in \{0, 1\}^{q(n)}}[M(x, r) = 1] \leq \frac{1}{2}$$

(a) Show that $NP \subseteq PP$

(b) Show that the problem $#SAT = \{(\phi, k) : \text{the formula } \phi \text{ has } > k \text{ satisfying assignments}\}$ is PP-complete for Karp reductions.