1. Consider the threshold function $Th_2(x_1, \ldots, x_n)$ defined to be 1 if and only if at least two of the input variables are 1.

Prove that $\text{size}_{\{\land, \lor, \neg\}}(Th_2) \leq 4n + O(1)$

Prove that $\text{size}_{B_2}(Th_2) \geq 3n - O(1)$, where $B_2$ is the full binary basis. (Hint: show that if two variables are inputs to some binary gate, then at least one of them must be used elsewhere in the circuit).

2. A counting argument shows that almost all $n$-input boolean functions have minimal circuit size $\Omega(2^n/n)$. Show that this bound can be matched. In other words, show that any boolean function can be computed by a bounded fan-in circuit of size $O(2^n/n)$.

Hint: You should treat the last $\log(n - \log n)$ bits separately from the remainder. Start by computing all possible functions on these last few bits. How many gates does this require? How can you use this to design a circuit for the original function? Remember that if you have a function $f(x_1, x_2, \ldots, x_n)$ then you can represent it as follows:

$$f(x_1, x_2, \ldots, x_n) = [x_1 \land f(1, x_2, \ldots, x_n)] \lor [(\neg x_1) \land f(0, x_2, \ldots, x_n)]$$

3. Show that the multiplication of two $n$-bit integers cannot be computed in $AC^0$.

4. Prove that the probability of having less than $\sqrt{n}/2$ stars assigned by a random restriction that assigns stars independently with probability $1/\sqrt{n}$ is $O(n^{-1/2})$.

5. A two-way probabilistic finite automata (2PFA) $A$ is a two-tape Turing machine with one input tape and another tape which holds a random bit string. $A$ has a read-only two-way access to the input tape and read-only one-way access to the random tape. So $A$ cannot write and hence the name 2PFA. We say that $A$ recognizes a language $L$ if $A$ halts with probability 1 on all the inputs, and if

$$x \in L \implies \Pr[A \text{ accepts } x] > 2/3$$

$$x \not\in L \implies \Pr[A \text{ accepts } x] < 1/3$$

Show that $\{0^n1^n : n \geq 0\}$ is recognizable by a 2PFA.