1. Prove that if $\text{NP} \subseteq \text{P}/\log$ then $\text{P} = \text{NP}$.

Here our TM has access to an advice string, an extra input to the Turing machine that is allowed to depend on the length $n$ of the original input, but not on the input itself. A decision problem is in the complexity class $\text{P}/f(n)$ if there is a polynomial time Turing machine $M$ such that for any $n$, there is an advice string $A$ of length $f(n)$ such that, for any input $x$ of length $n$, the machine $M$ correctly decides the problem on the input $x$.

2. Show that 2-SAT is $\text{NL}$ complete

3. (a) An $\text{NP}$-optimization problem is given by a polynomial-time computable objective function $\text{Obj} : \Sigma^* \times \Sigma^* \rightarrow \mathbb{Q}_{\geq 0}$, where $\text{Obj}(x, y) = +\infty$ if $|y| > p(|x|)$ for some polynomial $p$. The problem is: given an input $x$, find $y$ minimizing $\text{Obj}(x, y)$. An example is the problem of finding the shortest tour in an instance of the Travelling Salesman Problem. Prove that the following are equivalent:
   - $\text{NP} = \text{P}$
   - Every $\text{NP}$ search problem can be solved in polynomial time.
   - Every $\text{NP}$ optimization problem can be solved in polynomial time.

   (b) Show that a language $L$ is $\text{NP}$-complete if and only if $L^c$ is $\text{coNP}$-complete (with respect to poly-time mapping reductions).

   (c) Show that if $\text{NP} \neq \text{coNP}$ then $\text{NP} \neq \text{P}$

   (d) Let $\text{Tautology} := \{\phi : \phi$ is a boolean formula such that $\forall a, \phi(a) = 1\}$. Show that Tautology is $\text{coNP}$-complete.

4. Can you compute from $x_1, x_2, x_3$ the three values $\neg x_1, \neg x_2, \neg x_3$ using AND gates, OR gates, and at most 2 NOT gates?

5. A boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is called symmetric if its value does not change when permuting its input bits. We denote by $f \circ g$ the class of depth 2 circuits where the output gate is $f$ and the gates at the first level are $g$. For example, $\text{AND} \circ \text{MAJ}$ denotes the class of depth 2 circuits where the output gate is AND and the first level has only majority gates.

Show that any symmetric function can be computed by linear size $\text{MAJ} \circ \text{MAJ}$ circuits. You are allowed to use constants in your circuit.
6. Let \( C = (C_n) \) be a family of circuits constructed with binary \( AND \) and \( OR \) gates. Assume that \( C \) has polynomial size and the graph of each \( C_n \) is a tree. Show that the induced boolean function is actually in \( NC^1 \). In other words, show that the circuit can be modified to have \( O(\log n) \) depth. Hint: Divide and Conquer.