Question 1

Consider the following algorithm to detect cycles in an undirected graph $G = (V, E)$. Imagine that a father and son are travelling along the edges of $G$. The father sits at a vertex $v$ (for $v = 1, 2, \ldots, |V|$), while the son traverses the graph according to the so-called “cycle searching principle”:

- For every vertex $u \in G$, give all edges out from $u$ an order according to their end vertex. Thus we can say that $u$ has a first edge, second edge, and so on. Note that an edge $(u, v)$ may be the $i$th edge for $u$ but the $j$th edge for $v$, where $i \neq j$.

- Whenever entering a vertex $u$ of fanout $k$ along the $i$th edge of $u$, the son leaves through edge $i + 1$ (here $k + 1$ is taken to be $1$)

The father remembers the edge along which the son departed and sees if he comes back along the same edge. If, for every edge adjacent to $v$, the son does so, the father takes his son to vertex $v + 1$ (if $v < |V|$), or declares that $G$ has no cycle (if $v = |V|$). Otherwise, he declares that there must be a cycle. Prove that this algorithm terminates in finite time, and that it is correct. What is the space complexity of the algorithm?

Question 2

A language is called unary if every string in it is of the form $1^i$ (the string of $i$ ones) for some $i \geq 0$; in other words it is a subset of $\{1\}^*$. Show that if a unary language is NP-complete then $P = NP$. Prove that if every unary NP-language is in $P$ then $EXP = NEXP$.

Here, $EXP = \bigcup_{k \in \mathbb{N}} DTIME(2^{n^k})$, i.e. all the problems solvable in exponential time. $NEXP$ is defined similarly.

Question 3

Assume that $DTIME(n) = NTIME(n)$. Show that $P = NP$.

Question 4

Find a non-regular language that you can recognize in $DSPACE(loglog(n))$. (In fact, it’s possible to show that this bound is tight, and that every language in $DSPACE(o(loglog(n)))$ is regular).

Question 5

In the proof that there is no optimal time bound, we used the assumption that you can pad the encoding of a TM $M$ with an arbitrary number of 1’s. Explain why you need this assumption in the proof.