COMP/MATH 533: Algorithmic Game Theory	September 24, 2014
Lecture 5	
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NOTE: The content of these notes has not been formally reviewed by the lecturer. It is recommended that they are read critically.

1 Motivation

So far we have focused on social surplus maximization only. One of the reasons why we've focused on surplus maximization is because it appears often in real world scenarios. For example, when the government puts to auction wireless bands it is in the best interest of everyone that whoever win makes the best use of the new band. Also, in competitive environments, it's assumed that you should try to maximize surplus, otherwise one of your competitors will [6]. On the other hand, revenue maximization is also a very important topic and it will be the main topic of this notes.

2 Setup and assumptions - Bayesian Analysis Model

For the rest of this notes we will make the following assumptions that will allow us to derive an expression capable of maximizing expected revenue.

- We are in a single parameter environment
- v_i is drawn from a distribution function F_i (not necessarily identical) with density f_i on $[0, v_{max}]$
- F_i are independent and know to the mechanism designer. Note that the realizations $v_1, v_2, v_3, \cdots, v_n$ are still unknown

3 Revenue Optimal auctions

First of all, what do we mean by optimal? In a similar fashion, what types of mechanisms should we consider? We will focus only on the mechanisms that have an ***Equilibrium Dominant Strategy***, and out of that subset of mechanisms we want to find the one which revenue is the *highest* when it reaches an equilibrium.

Note that, even with this rules it is still somehow unclear how we can find such a mechanism. Luckily enough, we learned about the ***Revelation Principle*** which given an auction with a dominant strategy, will be able to find an equivalent auction in which truth telling is a dominant strategy. Using this principle, we can safely make some assumptions and most importantly have a sound definition of our revenue function

4 Expected revenue equals expected virtual welfare

Since we are only considering direct-revelation DSCI mechanisms, we can assume that bidders truthfully bid, i.e. $b = v$. Also, let $\mathbf{F}= F_1 \times F_2 \times F_3 \times ... \times F_n$ be a **n** dimensional vector of independent distribution functions.

Before beginning the derivation, let's look one more time at Myerson's Lemma [5] :

$$P(\vec{v}) = P(v_i, v_{-i}) = v_i \cdot x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(t, v_{-i})dt$$

We can now start working on our revenue function which is defined by:

$$\sum_{\vec{v}} \sum_{i}^{n} P_i(\vec{v}) \cdot \Pr_F(\vec{v})$$

Which by definition is the sum of the expected value [3] (in this case of a discrete r.v.) of $\vec{v}$ over **F**

$$\sum_{\vec{v}} \sum_{i}^{n} P_i(\vec{v}) \cdot \Pr_F(\vec{v}) = \mathbb{E}_{\vec{v} \sim F} \left[\sum_{i}^{n} P_i(\vec{v}) \right]$$

By linearity of expectation [4], we can take the sum out

$$\sum_{\vec{v}} \sum_{i}^{n} P_i(\vec{v}) \cdot \Pr_F(\vec{v}) = \sum_{i}^{n} \left[\mathbb{E}_{\vec{v} \sim F}[P_i(\vec{v})] \right]$$

And the by independence of the distribution function [2] in **F**, we can expand the term as following

$$\sum_{\vec{v}} \sum_{i}^{n} P_i(\vec{v}) \cdot \Pr_F(\vec{v}) = \sum_{i}^{n} \left[\mathbb{E}_{v_{-i} \sim F_{-i}} [\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})]] \right] \tag{1}$$

Let's now focus on the inner term, by definition of expected value [1] (this time it's a continuous r.v.) we can expand it to its integral form

$$\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})] = \int_0^{v_{max}} P_i(v_i, v_{-i}) \cdot f_i(v_i) dv_i$$

We can now replace the first term in the integral by an equivalent expression given by Myerson's Lemma

$$\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})] = \int_0^{v_{max}} [v_i \cdot x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(t, v_i)dt] \cdot f_i(v_i) dv_i$$

If we expand and then try to simplify the expression, we have

$$\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})] = \int_0^{v_{max}} v_i \cdot x_i(v_i, v_{-i}) \cdot f_i(v_i) dv_i - \int_0^{v_{max}} \int_0^{v_i} x_i(t, v_{-i})dt \cdot f_i(v_i) dv_i$$

Next, we swap the order of the integration in the second term.

$$\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})] = \int_0^{v_{max}} v_i \cdot x_i(v_i, v_{-i}) \cdot f_i(v_i) dv_i - \int_0^{v_{max}} x_i(t, v_{-i}) \int_t^{v_{max}} f_i(v_i) dv_i \; dt$$

Note that, by definition of density function, we have

$$F(t) = \int_0^t f(u) du,$$

thus,

$$1 - F(t) = \int_t^{v_{max}} f_i(v_i) dv_i$$

We can simplify and regroup some of the terms

$$\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})] = \int_0^{v_{max}} v_i \cdot x_i(v_i, v_{-i}) \cdot f_i(v_i) dv_i - \int_0^{v_{max}} x_i(t, v_{-i})(1 - F_i(t)) dt$$

Notice that, we can replace variable t with v_i in the second term and further simplify the formula.

$$\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})] = \int_0^{v_{max}} v_i \cdot x_i(v_i, v_{-i}) \cdot f_i(v_i) dv_i - \int_0^{v_{max}} x_i(v_i, v_{-i})(1 - F_i(v_i)) dv_i$$

$$\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})] = \int_0^{v_{max}} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] \cdot x_i(v_i, v_{-i}) \cdot f_i(v_i) dv_i$$

Let $\varphi_i(v_i)$ be a special term defined as follows that we will call "virtual value"

$$\varphi_i(v_i) := v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

We can now rewrite out expression as

$$\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})] = \int_0^{v_{max}} \varphi_i(v_i) \cdot x_i(v_i, v_{-i}) \cdot f_i(v_i) dv_i$$

So we have that

$$\mathbb{E}_{v_i \sim F_i}[P_i(\vec{v})] = \mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \cdot x_i(v_i, v_{-i})]$$

Now that we have a new form for this expression, we can plug it back into (1)

$$\sum_i^n [\mathbb{E}_{\vec{v} \sim F}[P_i(\vec{v})]] = \sum_i^n \left[\mathbb{E}_{v_{-i} \sim F_{-i}}[\mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i) \cdot x_i(v_i, v_{-i})]] \right]$$

$$\sum_i^n [\mathbb{E}_{\vec{v} \sim F}[P_i(\vec{v})]] = \sum_i^n [\mathbb{E}_{\vec{v} \sim F}[\varphi_i(v_i) \cdot x_i(\vec{v})]]$$

$$\sum_i^n [\mathbb{E}_{\vec{v} \sim F}[P_i(\vec{v})]] = \mathbb{E}_{\vec{v} \sim F} \left[\sum_i^n \varphi_i(v_i) \cdot x_i(\vec{v}) \right]$$

We finally have a fairly simple to understand term which is our expected virtual welfare. Maximizing expected revenue reduces to maximizing the expected virtual welfare which we just derived.

5 Two Uniform [0,1] Bidders Example

Two Bidders, One item

- Two bidders values are drawn i.i.d. from $U[0,1]$
- Vickrey with reserve at $1/2$

  Remark 1. *If the highest bidder is lower than* $1/2$*, no one wins*

  Remark 2. *If the highest bidder is at least* $1/2$*, he wins the item and pay* $max[1/2, theotherbiddersbid]$

- Revenue equals 5/12

Why is it optimal? Virtual value for v:

$$\varphi(v) = v - \frac{(1-F(v))}{f(v)} = v - \frac{(1-v)}{1} = 2v - 1$$

Optimizing expected virtual welfare equals optimizing virtual welfare on every bid profile. For any bid profile (v_1, v_2), the following allocation rule optimizes virtual welfare:

- $(\varphi(v_1), \varphi(v_2)) = (2v_1 - 1, 2v_2 - 1)$
- If $max[v_1, v_2] \geq 1/2$, give the item to the highest bidder
- Otherwise, $\varphi(v1), \varphi(v2) < 0$. Should not give it to either of the two

This allocation rule is monotone and virtual value is easy to calculate. The unique allocation rule shows that 1/2 is DSIC.

6 Optimal Auction

The goal is to find the monotone allocation rule that optimizes expected virtual welfare.

First we consider virtual welfare. What allocation rule optimizes expected virtual welfare? We should optimize virtual welfare on every bid profile v. The equation is

$$\max \sum_i x_i(v)\varphi_i(v_i), \quad s.t \sum_i x_i(v) \leq 1.$$

We call this the **Virtual Welfare-Maximizing Rule**.

Whether the Virtual Welfare-Maximizing Rule is monotone depends on the distribution.

Definition 1. ***Regular Distributions*** *A single-dimensional distribution* F *is regular if the corresponding virtual value function* $v - \frac{1-F(v)}{f(v)}$ *is non-decreasing (intuition: higher real value means higher virtual value).*

Definition 2. ***Monotone Hazard Rate (MHR)*** *A single-dimensional distribution* F *has Monotone Hazard Rate, if* $\frac{1-F(v)}{f(v)}$ *is non-increasing*

Examples:

- **MHR:** uniform, exponential and Gaussian distributions
- **Regular:**MHR and Power-law.
- **Irregular:**Multi-modal or distributions with very heavy tails.

When all the F_i's are regular, the Virtual Welfare-Maximizing Rule is monotone!

Two extentions (won't teach in class)
1. What if the distributions are irregular:

- Point-wise optimizing virtual welfare is not monotone
- Need to find the allocation rule that maximizes expected virtual welfare among all monotone ones.
- This can be done by "ironing" the virtual value functions to make them monotone, and at the same time preserving the virtual welfare. (consult Myerson's original paper or Section 3.3.5 of Jason Hartline's book)

2. If we don't restrict ourselves to DSIC mechanisms:

- Myerson's auction is optimal even amongst a much larger set of "Bayesian incentive compatible (BIC)" (essentially the largest set) mechanisms.
- For example, this also means First-price auction can not achieve higher revenue (at equilibrium) than Myerson's auction.

For further reading, consider

- Section 3.3.5 in "Mechanism Design and Approximation", book draft by Jason Hartline.
- "Optimal auction design", the original paper by Roger Myerson.

References

[1] Expectation and variance: Continous random variables. `https://math.dartmouth.edu/archive/m60s06/public_html/Lecture17.pdf`. Accessed: 2014-09-23.

[2] Expectation chapter 7. `http://dept.stat.lsa.umich.edu/~michaelw/STAT425/expttnsml.pdf`. Accessed: 2014-09-23.

[3] Expected value and variance. `http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/Chapter6.pdf`. Accessed: 2014-09-23.

[4] Ronitt Rubinfeld Albert R. Meyer. Missed expectations. `http://courses.csail.mit.edu/6.042/fall05/ln14.pdf`. Accessed: 2014-09-23.

[5] Roger B. Myerson. Optimal auction design. `http://www.eecs.harvard.edu/~parkes/cs286r/spring07/papers/myerson.pdf`. Accessed: 2014-09-24.

[6] Tim Roughgarden. Cs364a: Algorithmic game theory lecture 5: Revenue-maximizing auctions. `http://theory.stanford.edu/~tim/f13/l/l5.pdf`. Accessed: 2014-09-23.