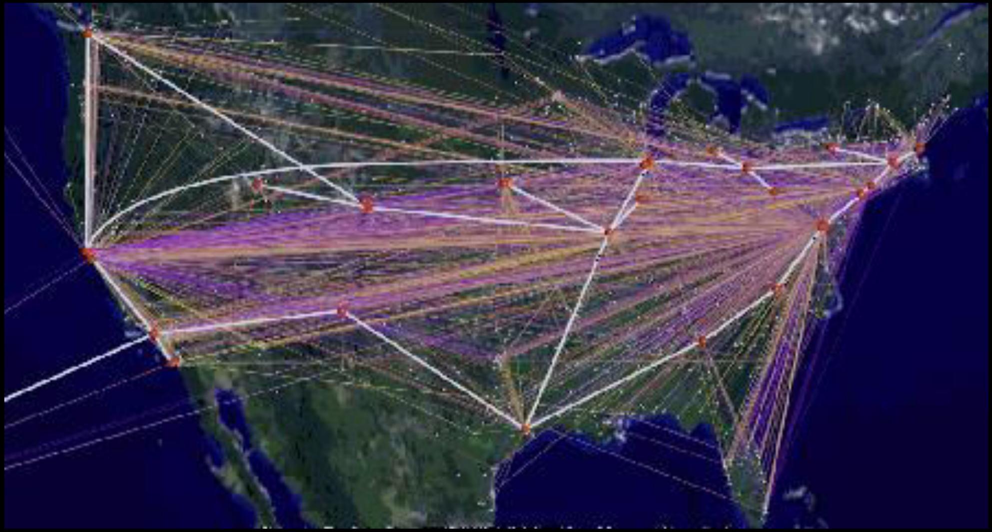
Comp553: Algorithmic Game Theory

Lecture 22

Fall 2014

Yang Cai

Markets



"Economics is a science which studies human behavior as a relationship between ends and scarce means which have alternative uses."

Lionel Robbins (1898 – 1984)

How are scarce resources assigned to alternative uses?


i'm in everything


CHRIS MADDEN

How are scarce resources assigned
to alternative uses?

Prices!

Parity between demand and supply

equilibrium prices

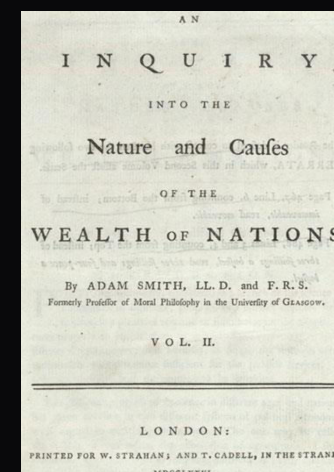
the beginnings of a mathematical theory

Adam Smith (1723-1790)

The Wealth of Nations (1776): the "invisible hand of the economy"

"By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it."

"It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from their regard to their own interest."



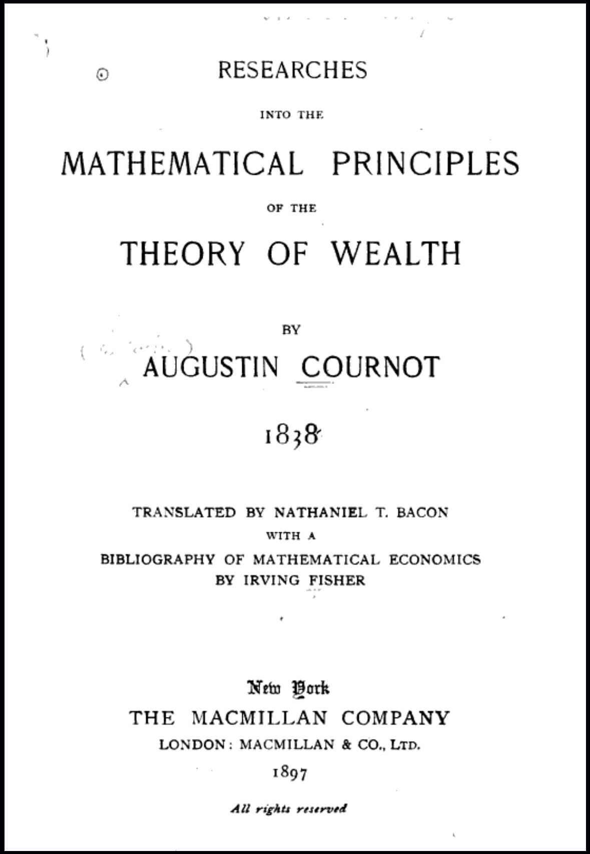
AN
INQUIRY
INTO THE
Nature and Caufes
OF THE
WEALTH OF NATIONS.

By ADAM SMITH, LL.D. and F.R.S.
Formerly Profeffor of Moral Philofophy in the Univerfity of Glasgow.

VOL. II.

LONDON:
PRINTED FOR W. STRAHAN; AND T. CADELL, IN THE STRAND.
MDCCLXXVI.

Augustin Cournot (1801-1877)



RESEARCHES

INTO THE

MATHEMATICAL PRINCIPLES

OF THE

THEORY OF WEALTH

BY

AUGUSTIN COURNOT

1838

TRANSLATED BY NATHANIEL T. BACON

WITH A

BIBLIOGRAPHY OF MATHEMATICAL ECONOMICS

BY IRVING FISHER

New York

THE MACMILLAN COMPANY

LONDON: MACMILLAN & CO., LTD.

1897

standpoint, a subject of general interest which has so many different sides.

But the title of this work sets forth not only theoretical researches; it shows also that I intend to apply to them the forms and symbols of mathematical analysis. This is a plan likely, I confess, to draw on me at the outset the condemnation of theorists of repute. With one accord they have set themselves against the use of mathematical forms, and it will doubtless be difficult to overcome to-day a prejudice which thinkers, like Smith and other more modern writers, have contributed to strengthen. The reasons for this prejudice seem to be, on the one hand, the false point of view from which theory has been regarded by the small number of those who have thought of applying mathematics to it; and, on the other hand, the false notion which has been formed of this analysis by men otherwise judicious and well versed in the subject of Political Economy, but to whom the mathematical sciences are unfamiliar.

Cournot's Contributions:

notion of a *demand function*

D = F(p), where p is the price; F(.) is assumed continuous, and it is taken as an empirical proposition that the demand function is decreasing

analysis of a *monopoly*:

- profit-maximizing producer with *cost* f(D), for production D; discusses decreasing, constant and increasing cost functions

- equations determining *equilibrium price*

duopoly model:

two rival producers of a homogeneous product

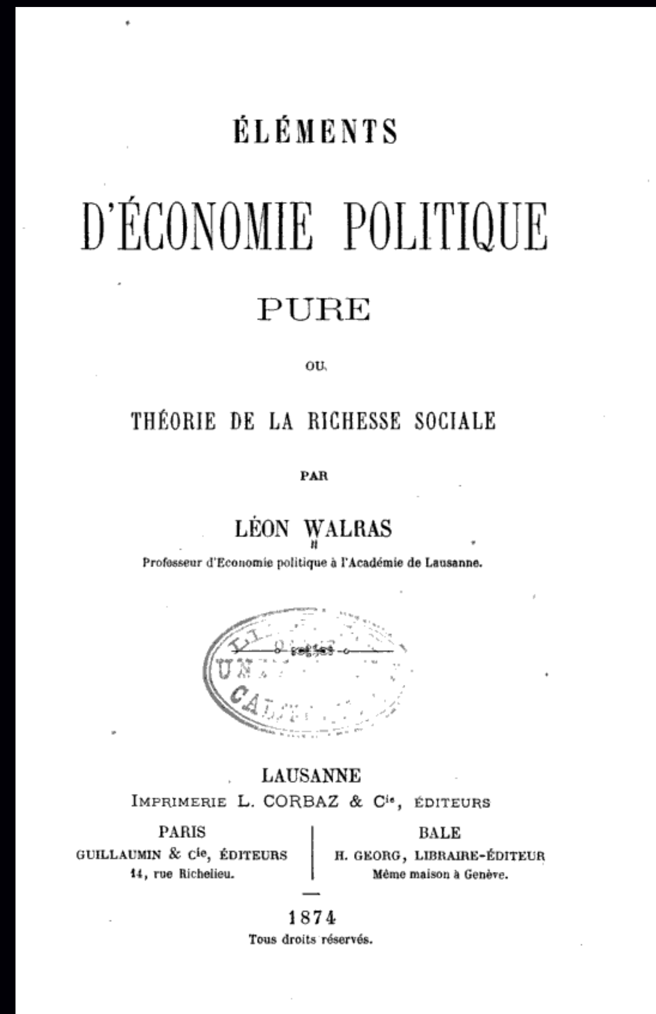
unlimited competition,

communication of markets on single commodity, ...

question left unanswered by Cournot…

react on each other. An increase in the income of the producers of commodity A will affect the demand for commodities B, C, etc., and the incomes of their producers, and, by its reaction, will involve a change in the demand for commodity A. It seems, therefore, as if, for a complete and rigorous solution of the problems relative to some parts of the economic system, it were indispensable to take the entire system into consideration. But this would surpass the powers of mathematical analysis and of our practical methods of calculation, even if the values of all the constants could be assigned to them numerically. The object

Leon Walras (1834-1910)



ÉLÉMENTS

D'ÉCONOMIE POLITIQUE

PURE

OU

THÉORIE DE LA RICHESSE SOCIALE

PAR

LÉON WALRAS

Professeur d'Economie politique à l'Académie de Lausanne.

LAUSANNE

IMPRIMERIE L. CORBAZ & Cie, ÉDITEURS

PARIS	BALE
GUILLAUMIN & Cie, ÉDITEURS	H. GEORG, LIBRAIRE-ÉDITEUR
14, rue Richelieu.	Même maison à Genève.

1874

Tous droits réservés.

Elements of Pure Economics, or the theory of social wealth, 1874

Leon Walras (1834-1910)

- goal was to solve the problem that was left open by Cournot, i.e. characterize the price equilibrium of a market with many commodities;

- gave system of simultaneous equations for price equilibrium:

  informal argument for the existence of an equilibrium based on the assumption that an equilibrium exists whenever the number of equations equals the number of unknowns

- recognized the need for *dynamics leading to an equilibrium*

  tâtonnement: price adjustment mechanism

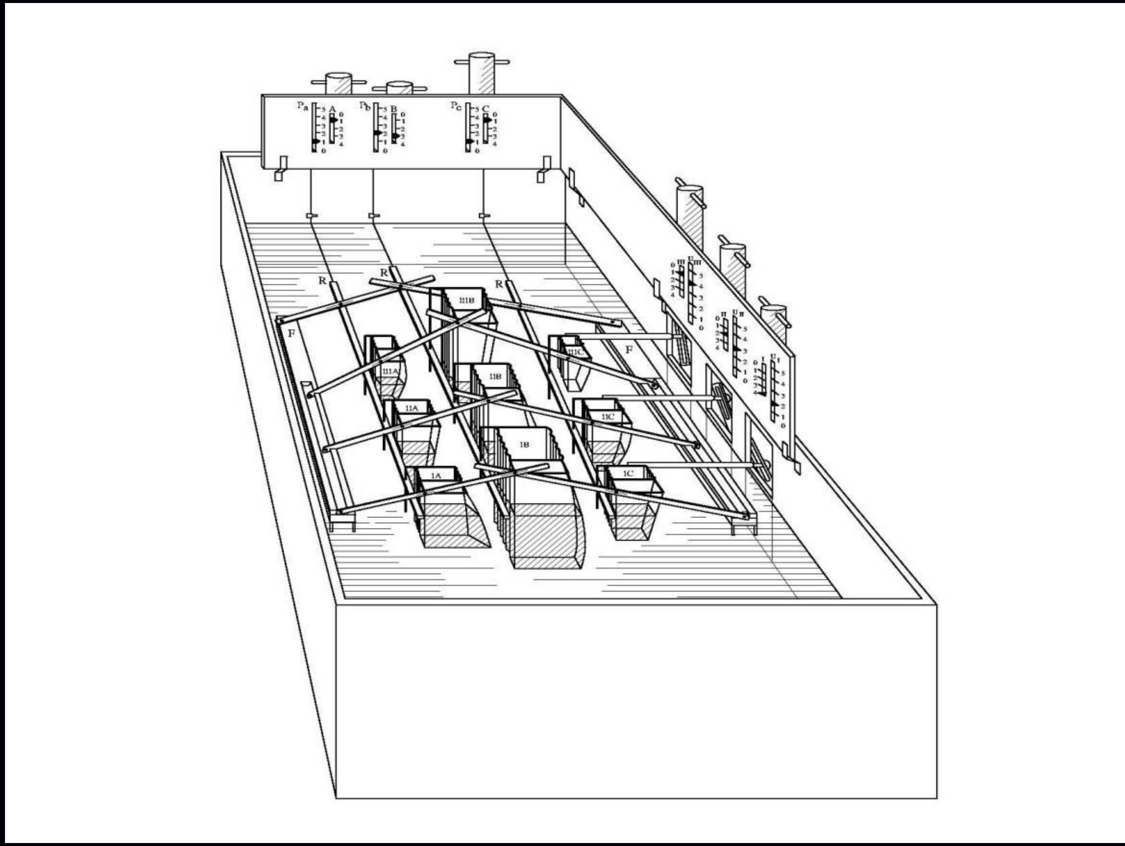
Irving Fisher (1867-1947)

hydraulic apparatus for solving markets with 3 traders with money and a producer of 3 commodities

Irving Fisher (1867-1947)

First computational approach!

1891



[Irving Fisher (1867-1947)

Stock Market Crash of 1929:

"Stock prices have reached what looks like a permanently high plateau."

[Shortly before the crisis]

Market is "only shaking out of the lunatic fringe"

[October 21, 1929 (8 days before black Tuesday)]

]

[....]

Arrow-Debreu-McKenzie Theorem

One of the most celebrated theorems in Mathematical Economics.

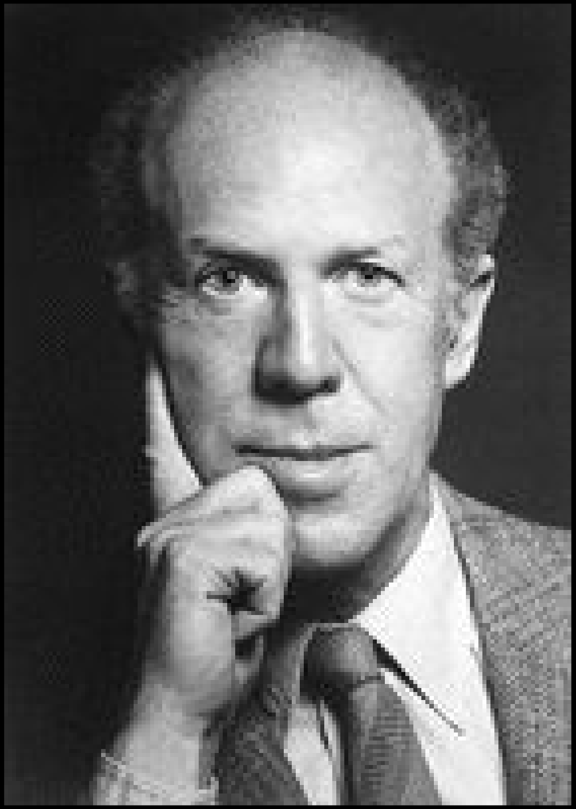
Established the existence of a market equilibrium under very general conditions using Brouwer/Kakutani's fixed point theorem.

it is hence highly non-constructive!

Kenneth Arrow

- Nobel Prize, 1972

Gerard Debreu



- Nobel Prize, 1983

Exchange Market Model (without production)

Consider a marketplace with:

n traders (or agents)

k goods (or commodities)
assumed to be *infinitely divisible*

Utility function of trader i:

$$u_i \; : \; \mathcal{X}_i \subseteq \mathbb{R}^k_+ \longrightarrow \mathbb{R}_+$$

non-negative reals

consumption set for trader i

specifies trader i's utility for bundles of goods

Endowment of trader i:

$$e_i \in \mathcal{X}_i$$

amount of goods trader comes to the marketplace with

Exchange Market Model (without production)

Suppose the goods in the market are priced according to some price vector $p \in \mathbb{R}^k_+$.

Under this price vector, each trader would like to sell some of her endowment and purchase an optimal bundle using her income from what s/he sold; thus she solves the following program:

$$\left.\begin{array}{rl} \max & u_i(x) \\ \text{s.t.} & p \cdot x \le p \cdot e_i \\ & x \in \mathcal{X}_i \end{array}\right\} \text{Program}_i(p)$$

Note: If u_i is continuous and $\mathcal{X}_i$ is compact, then the above program has a well-defined optimum value.

Competitive (or Walrasian) Market Equilibrium

Def: A price vector $p \in \mathbb{R}^k_+$ is called a *competitive market equilibrium* iff there exists a collection of optimal solutions $x_i(p)$ to Program$_i$(p), for all i = 1,…, n, such that the total supply meets the total demand, i.e.

$$\underbrace{\sum_{i=1}^{n} x_i(p)}_{\text{total demand}} \leq \underbrace{\sum_{i=1}^{n} e_i}_{\text{total supply}}$$

Arrow-Debreu Theorem 1954

Theorem [Arrow-Debreu 1954]: Suppose

(i) $\mathcal{X}_i$ is closed and convex

(ii) $e_i >> 0,\ \text{for all}\ i$ (all coordinates positive)

(iii a) u_i is continuous

(iii b) u_i is quasi-concave

$$u_i(x) > u_i(y) \implies u_i(\lambda x + (1-\lambda)y) > u_i(y),\ \forall \lambda \in (0,1)$$

(iii c) u_i is nonsatiated

$$\forall y \in \mathcal{X}_i,\quad \exists x \in \mathcal{X}_i \text{ s.t. } u_i(x) > u_i(y)$$

Then a competitive market equilibrium exists.

Market Clearing

Nonsatiation + quasi-concavity

- ➔ at equilibrium every trader spends all her budget, i.e. if $x_i(p)$ is an optimal solution to Program$_i(p)$ then

$$p \cdot x_i(p) = p \cdot e_i$$

$$\implies p \cdot \left(\sum_i x_i(p) - \sum_i e_i \right) = 0$$

- ➔ every good with positive price is fully consumed

A market with no equilibrium

Alice has oranges and apples, but only likes apples.

Bob only has oranges, and likes both oranges and apples.

- if oranges are priced at 0, then Bob's demand is not well-defined.

- if oranges are priced at > 0, then Alice wants more apples than there are in the market.

Proof of the Arrow-Debreu Theorem

Steps (details on the board)

simplifying assumption: u_i is strictly concave

(i) w.l.o.g. can assume that the $\mathcal{X}_i$ are compact

argument on the board; the idea is that we can replace $\mathcal{X}_i$ with

$$\mathcal{X}_i \cap \{x \leq 2 \sum_i e_i\}$$

without missing any equilibrium, and without introducing spurious ones (can argue the optimal solution is in the interior of the new consumption set, thus the optimal solution is the same if we use the original $\mathcal{X}_i$)

(ii) by compactness and strict concavity:

for all p, there exists a unique maximizer $x_i(p)$ of Program$_i(p)$

(iii) by the maximum theorem: $x_i(p)$ is continuous on p

(iv) *rest of the argument on the board*

Utility Functions

Linear utility function (goods are perfect substitutes)

$$u_i(x) = \sum_j a_{ij} x_j$$

Leontief (or fixed-proportion) utility function

$$u_i(x) = \min_j \{a_{ij} x_j\}$$

e.g. buying ingredients to make a cake

e.g. rate allocation on a network

Cobb-Douglas utility function

$$u_i(x) = \prod_j x_j^{a_{ij}}, \quad \text{where } \sum_j a_{ij} = 1$$

Interpretation: the per-unit fraction of buyer *i*'s income used in purchasing good *j* is proportional to a_{ij}. i.e. at optimality $x_{ij}\, p_j = c\, a_{ij}$.

Utility Functions

CES utility functions:

$$u_i(x) = \left(\sum_j u_{ij} \cdot x_j^{\rho} \right)^{\frac{1}{\rho}}, \quad -\infty < \rho \leq 1$$

Convention: - If u_{ij}=0, then the corresponding term in the utility function is always 0.

- If $u_{ij} > 0$, x_j=0, and ρ<0, then $u_i(x)$=0 no matter what the other x_j's are.

$\rho = 1$ → linear utility form

$\rho \to -\infty$ → Leontief utility form

$\rho \to 0$ → Cobb-Douglas form

elasticity of substitution: $\sigma = \dfrac{1}{1-\rho}$

Fisher's Model

Suppose all endowment vectors are parallel…

$$e_i = m_i \cdot e, \quad m_i > 0, \quad m_i : \ \text{scalar}, e : \ \text{vector}$$

➔ relative incomes of the traders are independent of the prices.

Equivalently, we can imagine the following situation:

n traders, with specified money m_i

k divisible goods owned by seller; seller has q_j units of good j

Arrow-Debreu Thm ➔

(under the Arrow-Debreu conditions) there exist prices that the seller can assign on the goods so that the traders spend all their money to buy optimal bundles and supply meets demand

Fisher's Model with CES utility functions

$$u_i(x_i) = \left(\sum_j u_{ij} \cdot x_{ij}^{\rho} \right)^{\frac{1}{\rho}}, \quad -\infty < \rho \le 1$$

- Buyers' optimization program (under price vector *p*):

$$\begin{aligned} \max \quad & u_i(x_i) \\ \text{s.t.} \quad & \sum_j x_{ij} p_j \le m_i \end{aligned}$$

- Global Constraint:

$$\sum_i x_{ij} \le q_j, \quad \forall j$$

$$x_{ij} \ge 0, \quad \forall j$$

Eisenberg-Gale's Convex Program

- The space of feasible allocations is:

$$\sum_i x_{ij} \leq q_j, \quad \forall j$$
$$x_{ij} \geq 0, \quad \forall j$$

- But how do we aggregate the trader's optimization problems into one global optimization problem?

e.g., choosing as a global objective function the sum of the traders' utility functions won't work…