COMP/MATH 553 Algorithmic Game Theory
Lecture 12: Implementation of the Reduced Forms and the Structure of the Optimal Multi-item Auction

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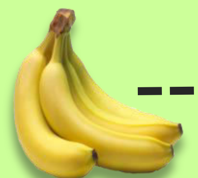
New Decision Variables

Variables: Interim Allocation rule aka. **"REDUCED FORM"**:

$$\{\pi_i : T_i \longrightarrow [0,1]^n, p_i : T_i \longrightarrow \mathbb{R}^+\}_{i \in [m]}$$

* $\pi_{ij}(v_i)$: $\underset{\vec{t}_{-i} \sim \mathcal{D}_{-i}}{\mathbf{Pr}}$ (j  →  i |  i valuation v_i)

* $\hat{p}_i(v_i)$: $\underset{\vec{t}_{-i} \sim \mathcal{D}_{-i}}{\mathbf{E}}$ [$price_i$ |  i valuation v_i]

A succinct LP

- Variables:
  - $\pi_{ij}(v_i)$: probability that item ***j*** is allocated to bidder ***i*** if her reported valuation *(bid)* is v_i in expectation over every other bidders' valuations (bids);
  - $p_i(v_i)$: price bidder ***i*** pays if her reported valuation (bid) is v_i in expectation over every other bidder's valuations (bids)
- Constraints:
  - BIC: $\sum_j v_{ij} \cdot \pi_{ij}(v_i) - p_i(v_i) \geq \sum_j v_{ij} \cdot \pi_{ij}(v_i') - p_i(v_i')$ for all v_i *and* v'_i in T_i
  - IR: $\sum_j v_{ij} \cdot \pi_{ij}(v_i) - p_i(v_i) \geq 0$ for all v_i *in* T_i
  - Feasibility: exists an auction with this reduced form.
- Objective:
  - Expected revenue: $\sum_i \sum_{v_i \in T_i} \Pr[t_i = v_i] \cdot p_i(v_i)$

Implementation of a Feasible Reduced Form

- After solving the succinct LP, we find the optimal reduced form π^* and p^*.

- **Can you turn π^* and p^* into an auction whose reduced form is exactly π^* and p^*?**

- This is crucial, otherwise being able to solve the LP is meaningless.

- Will show you a way to implement any feasible reduced form, and it reveals important structure of the revenue-optimal auction!

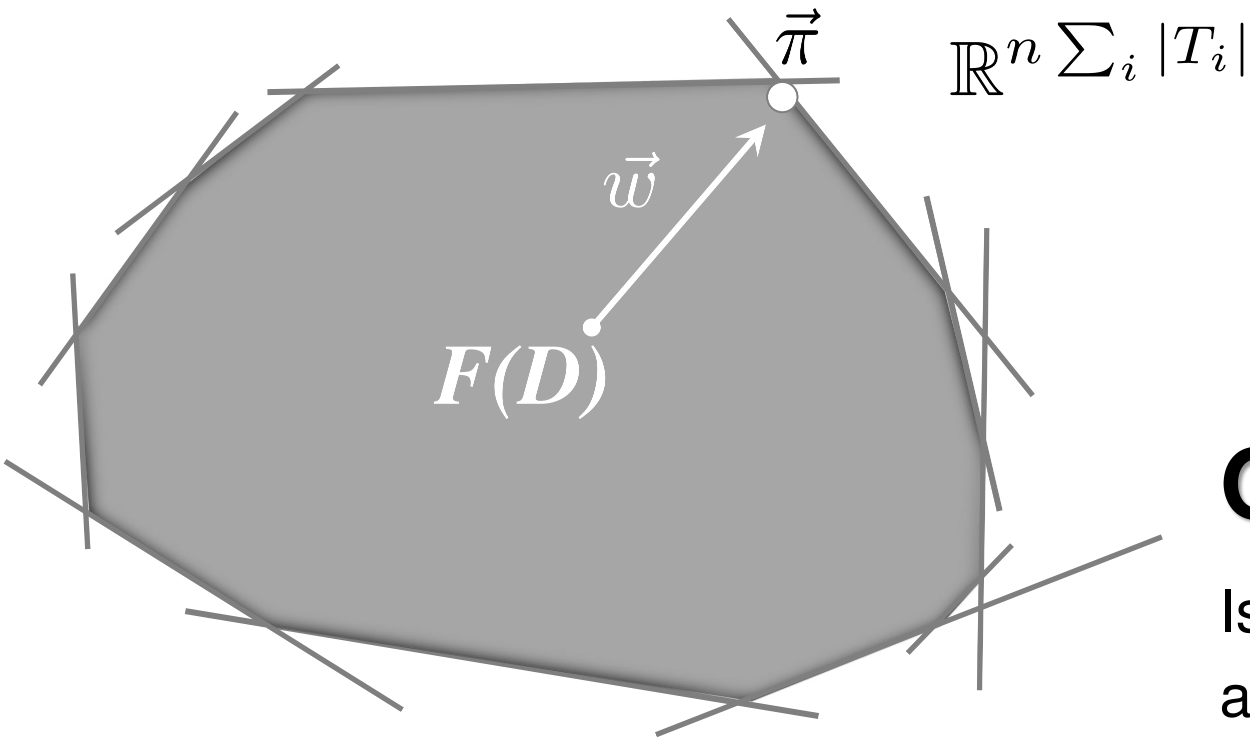
Implementation of a Feasible Reduced Form

Set of *Feasible* Reduced Forms

- Reduced form is collection $\{\pi_i : T_i \longrightarrow [0,1]^n\}$;
- Can view it as a vector $\vec{\pi} \in \mathbb{R}^{n \sum_i |T_i|}$;
- Let's call set of feasible reduced forms $F(D) \in \mathbb{R}^{n \sum_i |T_i|}$;

- **Claim 1: *F(D)* is a convex polytope.**
- **Proof:** ***Easy!***
  - A feasible reduced form $\vec{\pi}$ is implemented by a feasible allocation rule ***M***.
  - ***M*** is a distribution over deterministic feasible allocation rules, of which there is a finite number. So: $M = \sum_{\ell=1}^{k} p_\ell \cdot M_\ell$, where M_ℓ is **deterministic**.
  - Easy to see: $\vec{\pi} = \sum_{\ell=1}^{k} p_\ell \cdot \vec{\pi}(M_\ell)$
  - So, ***F(D)*** = $\left(\text{convex hull of reduced forms of feasible deterministic mechanisms} \right)$

Set of *Feasible* Reduced Forms



Q:

Is there a **simple** allocation rule implementing the corners?

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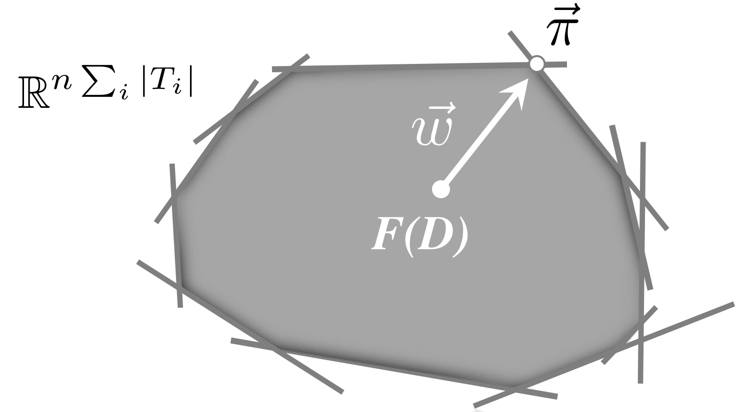


virtual welfare maximizing interim rule when virtual value functions are the f_i's

$$\vec{\pi} \in! \text{argmax}_{\vec{\pi}' \in F(D)} \{\vec{\pi}' \cdot \vec{w}\}$$

expected **virtual** welfare of an allocation rule with interim rule π'

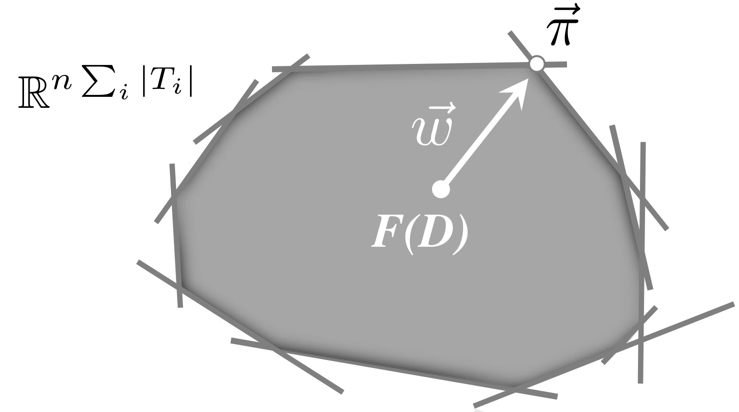
$$\vec{\pi}' \cdot \vec{w} = \sum_i \sum_j \sum_{A \in T_i} \pi'_{ij}(A) w_{ij}(A) \quad \text{------ (1)}$$

$$= \sum_i \sum_j \sum_{A \in T_i} \pi'_{ij}(A) f_{ij}(A) \Pr[t_i = A] \quad \text{--- (2)}$$

interpretation: **virtual** value derived by bidder *i* when given item *j* when his type is *A*

$$f_{ij}(A) := \frac{w_{ij}(A)}{\Pr_{\mathcal{D}}[t_i = A]}$$

Is there a simple allocation rule implementing a corner?



virtual welfare maximizing interim rule when virtual value functions are the f_i's

$$\vec{\pi} \in! \mathrm{argmax}_{\vec{\pi}' \in F(D)} \{\vec{\pi}' \cdot \vec{w}\}$$

?

Q: Can you name an algorithm doing this?

A: YES, the VCG allocation rule (w/ virtual value functions f_i, i=1,..,m) $=:$ virtual-VCG($\{ f_i \}$)

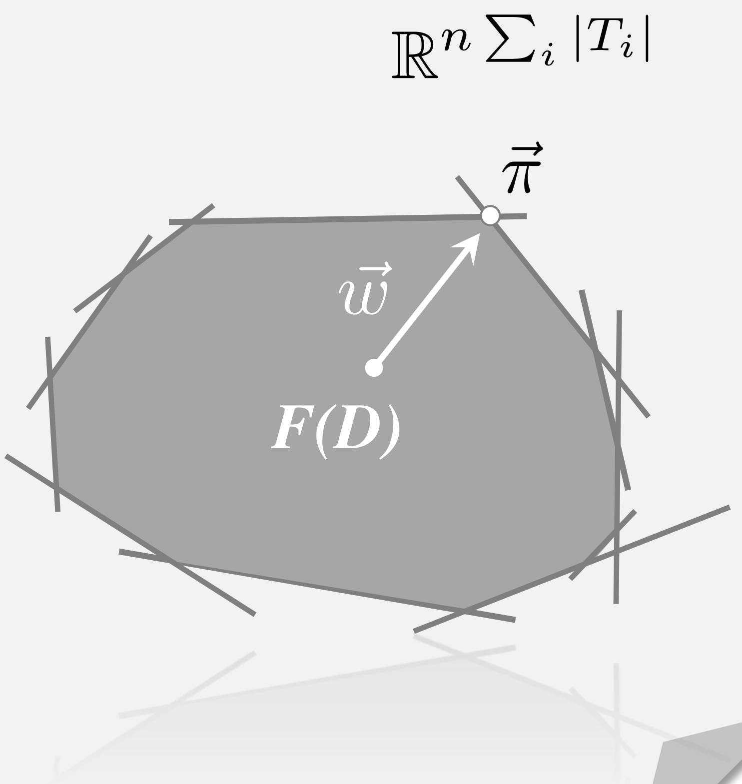
interpretation: **virtual** value derived by bidder i when given item j when his type is A

$$f_{ij}(A) := \frac{w_{ij}(A)}{\Pr_{\mathcal{D}}[t_i = A]}$$

Characterization Theorem [C.-Daskalakis-Weinberg]

F(D) is a **Convex Polytope** whose corners are **implementable** by **virtual VCG** allocation rules.

How about implementing any point inside ***F(D)***?



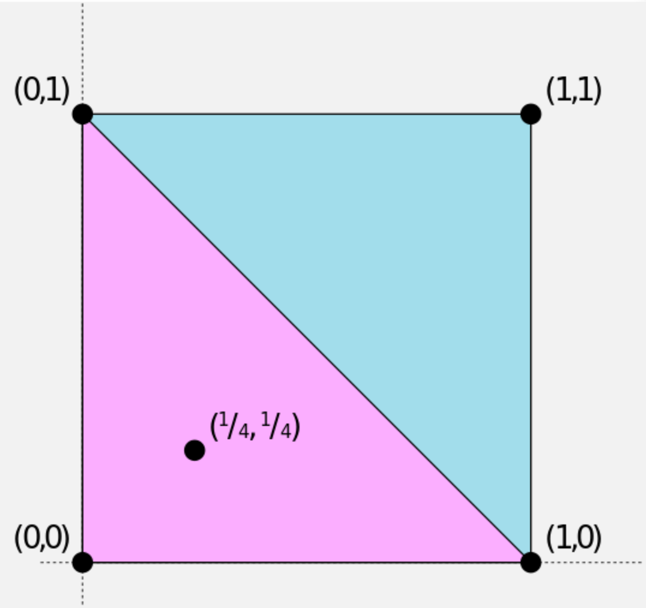
Carathéodory's theorem

If some point x is in the convex hull of P then

$$x = \sum_{p_i \in P} q_i \cdot p_i$$

$$\text{s.t.} \sum_i q_i = 1 \text{ and } q_i \geq 0 \ \forall i$$

Carathéodory's Theorem: If a point x of R^d lies in the convex hull of a set P, there is a subset P′ of P consisting of d + 1 or fewer points such that x lies in the convex hull of P′.



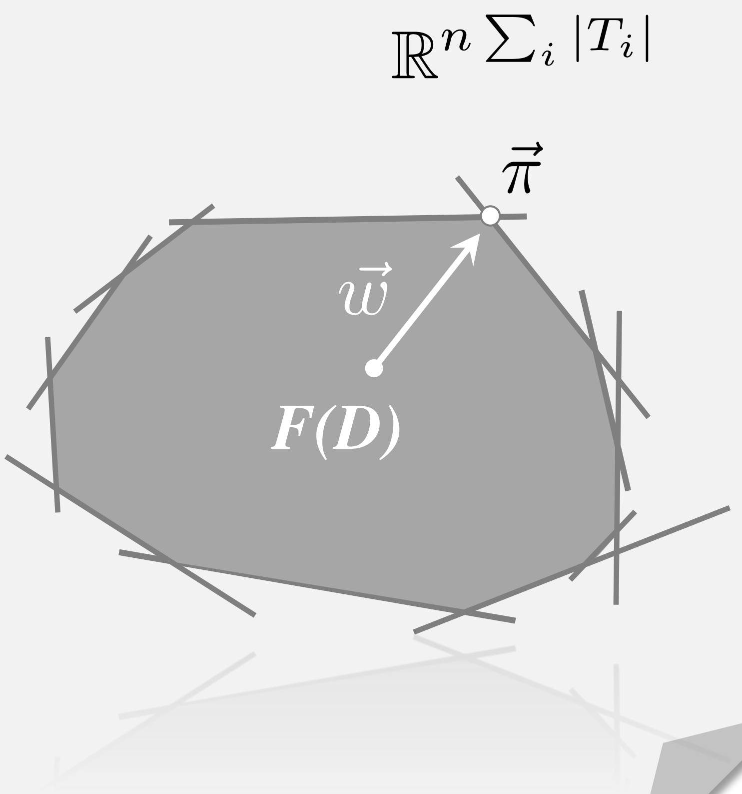
For example:

x = ¼(0,1) + ¼(1,0) + ½(0,0)

Characterization Theorem [C.-Daskalakis-Weinberg]

- Any point inside ***F(D)*** is a convex combination (distribution) over the corners.

- The interim allocation rule of any feasible mechanism can be implemented as **a distribution over virtual VCG allocation rules**.



Structure of the Optimal Auction

Characterization of Optimal Multi-Item Auctions

Theorem [C.-Daskalaks-Weinberg]: Optimal multi-item auction has the following structure:

1. Bidders submit valuations $(t_1,\ldots,t_m)$ to auctioneer.
2. Auctioneer samples virtual transformations $f_1,\ldots, f_m$
3. Auctioneer computes virtual types $t'_i = f_i(t_i)$
4. Virtual welfare maximizing allocation is chosen.

   Namely, each item is given to bidder with highest virtual value for that item (if positive)
5. Prices are charged to ensure truthfulness.

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- Exact same structure as Myerson!
  - in Myerson's theorem: virtual function = deterministic
  - here, ***randomized*** (and they must be)

Interesting Open Problems

- Another difference: in Myerson's theorem: virtual function is given explicitly, in our result, the transformation is computed by an LP. Is there any structure of our transformation?

- In single-dimensional settings, the optimal auction is DSIC. In multi-dimensional settings, this is unlikely to be true. What is the gap between the optimal BIC solution and the optimal DSIC solution?