COMP/MATH 553 Algorithmic Game Theory
Lecture 11: Revenue Maximization in Multi-Dimensional Settings

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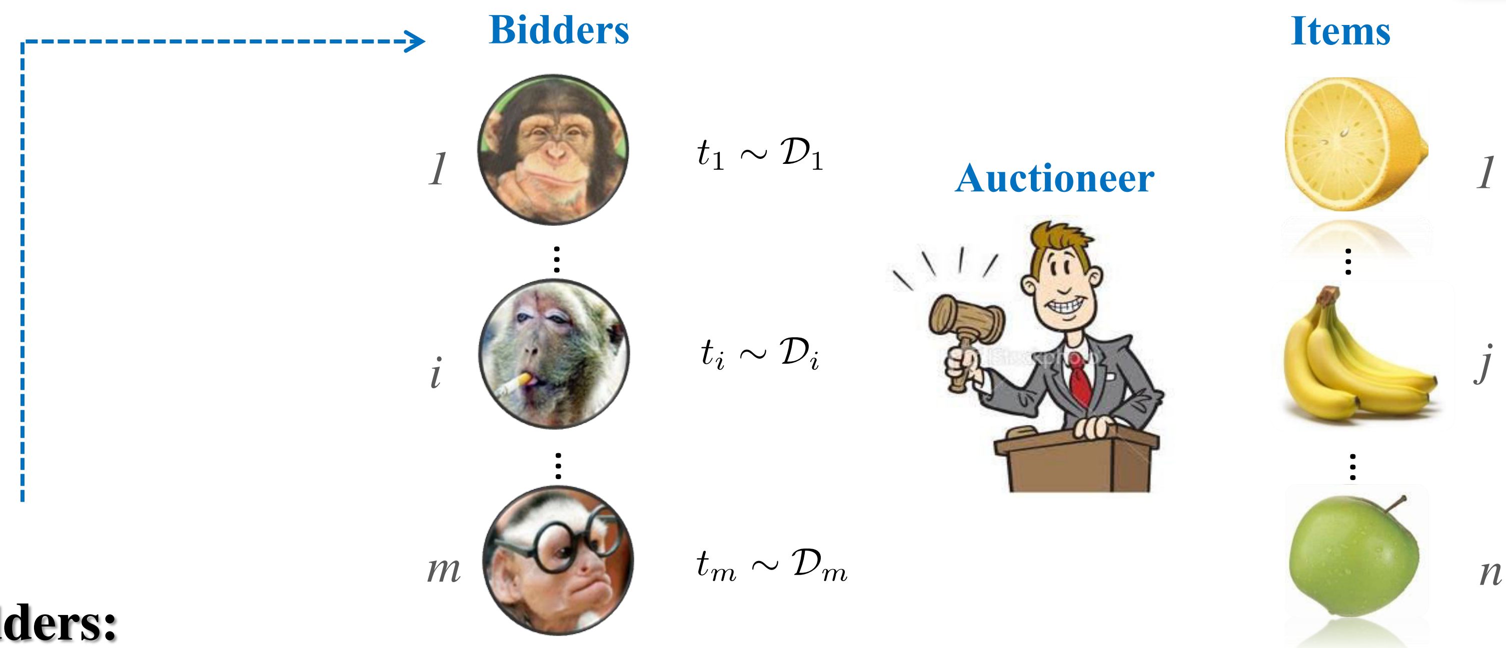
An overview of today's class

- *Basic LP Formulation for Multiple Bidders*
- *Succinct LP: Reduced Form of an Auction*
- *The Structure of the Optimal Auction*

Multi-item Multi-bidder Auctions: Set-up



Bidders:

- have values on "items" and bundles of "items".
- ***Valuation*** aka ***type*** $t_i \in T_i$ encodes that information.
- **Common Prior:** Each t_i is sampled independently from $\mathcal{D}_i$.
  - Every bidder and the auctioneer knows $\mathcal{D}$
- **Additive:** Values for bundles of items = sum of values for each item.
  - **From now on,** $t_i = (v_{i1}, \ldots, v_{in})$.

Basic LP Formulation: single bidder

- Variables:
  - Allocation rule: for each item $\boldsymbol{j}$ in $[\boldsymbol{n}]$, each valuation $\boldsymbol{v}$ in $\boldsymbol{T}$, there is a variable $\boldsymbol{x_j(v)}$: the probability that the buyer receives item $\boldsymbol{j}$ when his report is $\boldsymbol{v}$.
  - Payment rule: for each valuation $\boldsymbol{v}$ in $\boldsymbol{T}$, there is a variable $\boldsymbol{p(v)}$: the payment when the bid is $\boldsymbol{v}$.
- Objective function: $\max \Sigma_v \Pr[t = v]\, p(v)$
- Constraints:
  - incentive compatibility: $\Sigma_j v_j\, x_j(v) - p(v) \geq \Sigma_j v_j\, x_j(v') - p(v')$ for all $\boldsymbol{v}$ and $\boldsymbol{v'}$ in $\boldsymbol{T}$
  - individual rationality (non-negative utility): $\Sigma_j v_j\, x_j(v) - p(v) \geq 0$ for all $\boldsymbol{v}$ in $\boldsymbol{T}$
  - feasibility: $0 \leq x_j(v) \leq 1$ for all $\boldsymbol{j}$ in $[\boldsymbol{n}]$ and $\boldsymbol{v}$ in $\boldsymbol{T}$

Single Bidder Case

- Once the LP is solved, we immediately have a mechanism.
- Let x* and p* be the optimal solution of our LP. Then when the bid is $\boldsymbol{v}$, give the buyer item $\boldsymbol{j}$ with prob. $\boldsymbol{x_j^*(v)}$ and charge him $\boldsymbol{p^*(v)}$.
- How long does it take to solve this LP?
- # of variables = **(n+1)|T|**; # of constraints = **$|T|^2$+2n|T|**
- Both are polynomial in **n** and **|T|** (input size), we can solve this LP in time polynomial in the input size!

Multiple Bidders setting

- $\boldsymbol{m}$ bidders and $\boldsymbol{n}$ items. All bidders are additive.

- T_i is the set of possible valuations of bidder $\boldsymbol{i}$. It's a subset of R^n.

- Random variable t_i in R^n represents i's valuation. We assume t_i is drawn independently from distribution D_i, whose support is T_i.

- We know $\mathbf{Pr}[t_i = v_i]$ for every v_i in T_i and $\Sigma_v \mathbf{Pr}[t_i = v_i] = 1$.

- Some notations:
  - $T = T_1 \times T_2 \times \ldots \times T_m$
  - $D = D_1 \times D_2 \times \ldots \times D_m$
  - $t = (t_1, t_2, \ldots, t_m)$

Multiple Bidders: LP variables and objective

- Allocation Rule: for every bidder $\boldsymbol{i}$ in $[\boldsymbol{m}]$, every item $\boldsymbol{j}$ in $[\boldsymbol{n}]$, every valuation profile $\boldsymbol{v= (v_1, v_2, \ldots, v_m)}$ in $\boldsymbol{T}$, there is a variable $\boldsymbol{x_{ij}(v)}$: the probability that the buyer $\boldsymbol{i}$ receives item $\boldsymbol{j}$ when the reported valuation profile is $\boldsymbol{v}$ (bidder $\boldsymbol{i}$ reports $\boldsymbol{v_i}$).

- Payment Rule: for every bidder $\boldsymbol{i}$ in $[\boldsymbol{m}]$, every valuation profile $\boldsymbol{v}$ in $\boldsymbol{T}$, there is a variable $\boldsymbol{p_i(v)}$: the payment when the reported valuation profile is $\boldsymbol{v}$.

- Objective Function: $\textbf{max } \Sigma_{v \text{ in } T} \textbf{Pr}_{t\sim D}[t = v]\ \Sigma_i p_i(v)$

Multiple Bidders: LP Constraints

- With multiple bidders, there are two kinds of Incentive Compatibility
  - ***DSIC***
    - $\Sigma_j v_{ij} x_{ij}(v) - p_i(v) \geq \Sigma_j v_j x_{ij}(v'_i, v_{-i}) - p_i(v'_i, v_{-i})$ for every ***i***, every $\boldsymbol{v_i}$ and $\boldsymbol{v'_i}$ in $\boldsymbol{T_i}$ and $\boldsymbol{v_{-i}}$ in $\boldsymbol{T_{-i}}$
  - ***Bayesian Incentive Compatible (BIC)***
    - If every one else is bidding her true valuation, bidding my own true valuation is the optimal strategy.
    - If everyone is bidding truthfully, we have a Nash equilibrium.
    - For every ***i***, every $\boldsymbol{v_i}$ and $\boldsymbol{v'_i}$ in $\boldsymbol{T_i}$

$$\sum_{v_{-i} \in T_{-i}} \Pr[t_{-i} = v_{-i}] \Big(\sum_j v_{ij} x_{ij}(v) - p_i(v) \Big) \geq \sum_{v_{-i} \in T_{-i}} \Pr[t_{-i} = v_{-i}] \Big(\sum_j v_{ij} x_{ij}(v'_i, v_{-i}) - p_i(v'_i, v_{-i}) \Big)$$

Multiple Bidders: LP Constraints

- Similarly, we use the interim individual rationality (this doesn't make much difference)
  - If every one else is bidding her true valuation, bidding my own true valuation always give me non-negative utility.
  - For every ***i***, every $\boldsymbol{v_i}$ in $\boldsymbol{T_i}$

$$\sum_{v_{-i}\in T_{-i}} \Pr[t_{-i}=v_{-i}]\Big(\sum_j v_{ij}x_{ij}(v)-p_i(v)\Big)\geq 0$$

- Finally, the feasibility constraint
  - Since each item can be allocated to at most one bidder, we have the following
  - For all item ***j*** in *[**n**]* and valuation profile $\boldsymbol{v}$ in $\boldsymbol{T}$: $\Sigma_i x_{ij}(v) \leq 1$

Multiple bidders: Implementation

- Let x* and p* be the optimal solution of our LP. Then when the bid is $\boldsymbol{v}$, give the bidder $\boldsymbol{i}$ item $\boldsymbol{j}$ with prob. $\boldsymbol{x_{ij}^*(v)}$ and charge him $\boldsymbol{p_i^*(v)}$.

- How long does it take to solve this LP?

- What is the input size? Polynomial in **m**, **n** and $\boldsymbol{\Sigma_i |T_i|}$.

- # of variables = $\boldsymbol{(n+1)|T| = (n+1)\, \Pi_i\, |T_i|}$ (scales exponentially with the input)

- # of constraints = $\Sigma_i |T_i|^2 + \boldsymbol{2n|T|} = \Sigma_i |T_i|^2 + \boldsymbol{2n\, \Pi_i\, |T_i|}$ (again scales exponentially with the input)

- Takes *exponential time to even write down*, not mention solving it!!!

Any Solution for Multiple bidders?

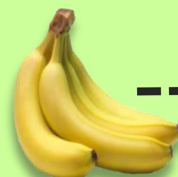
- The LP we discussed will only be useful if you have a small number of bidders.
- Is there a more succinct LP for our problem: polynomial in the size of the input.
- This is not only meaningful computationally.
- A more succinct LP in fact provides **conceptually insights** about the structure of the optimal mechanism in multi-item settings.

A New Succinct LP Formulation

New Decision Variables

Variables: Interim Allocation rule aka. **"REDUCED FORM"**:

$$\{\pi_i : T_i \longrightarrow [0,1]^n, p_i : T_i \longrightarrow \mathbb{R}^+\}_{i\in[m]}$$

* $\pi_{ij}(v_i)$: $\Pr_{\vec{t}_{-i} \sim \mathcal{D}_{-i}}$ (j  --->  i |  i valuation v_i)

* $\hat{p}_i(v_i)$: $\mathbf{E}_{\vec{t}_{-i} \sim \mathcal{D}_{-i}}$ [$price_i$ |  i valuation v_i]

Example of a reduced form

- Example: Suppose 1 item, 2 bidders

- Consider auction that allocates item preferring A to C to B to D, and charges \$2 dollars to whoever gets the item.

- For comparison: $x_{11}(A,C) = 1$, $x_{11}(A,D) = 1$, $x_{11}(B,C) = 0$ and $x_{11}(B,D) = 1$

- The reduced form: $\pi_{11}(A) = x_{11}(A,C) \times 0.5 + x_{11}(A,D) \times 0.5 = 1$;

  $p_1(A) = 2\times 0.5 + 2\times 0.5 = 2$

- Similarly, we can compute $\pi_{11}(B) = 1/2$, $\pi_{21}(C) = 1/2$, $\pi_{21}(D) = 0$;

  $p_1(B) = 1$, $p_2(C) = 1$ and $p_2(D) = 0$.

A succinct LP

- Variables:
  - $\pi_{ij}(v_i)$: probability that item $\boldsymbol{j}$ is allocated to bidder $\boldsymbol{i}$ if her reported valuation *(bid)* is v_i in expectation over every other bidders' valuations (bids);
  - $p_i(v_i)$: price bidder $\boldsymbol{i}$ pays if her reported valuation (bid) is v_i in expectation over every other bidder's valuations (bids)
- Constraints:
  - BIC: $\sum_j v_{ij} \cdot \pi_{ij}(v_i) - p_i(v_i) \geq \sum_j v_{ij} \cdot \pi_{ij}(v_i') - p_i(v_i')$ for all v_i *and* v'_i in T_i
  - IR: $\sum_j v_{ij} \cdot \pi_{ij}(v_i) - p_i(v_i) \geq 0$ for all v_i *in* T_i
  - Feasibility: exists an auction with this reduced form. Unclear?
- Objective:
  - Expected revenue: $\sum_i \sum_{v_i \in T_i} \Pr[t_i = v_i] \cdot p_i(v_i)$

Feasibility of Reduced Forms (example)

- Easy setting: **single item**, two bidders with types uniformly distributed in $T_1=\{A, B, C\}$ and $T_2=\{D, E, F\}$ respectively

- Question: Is the following interim allocation rule feasible?

bidder 1 $\xrightarrow{1/3}$ A $\pi_{11}(A) = 1$ $\qquad$ $\pi_{21}(D) = 2/3$ D $\xleftarrow{1/3}$ bidder 2

bidder 1 $\xrightarrow{1/3}$ B $\pi_{11}(B) = 0.5$ $\qquad$ $\pi_{21}(E) = 5/9$ E $\xleftarrow{1/3}$ bidder 2

bidder 1 $\xrightarrow{1/3}$ C $\pi_{11}(C) = 0$ $\qquad$ $\pi_{21}(F) = 0$ F $\xleftarrow{1/3}$ bidder 2

INFEASIBLE!

(A, D/E/F) → A wins. $\pi_{11}(A) = 1$ ✔

(B/C, D) → D wins. $\pi_{21}(D) = 2/3$ ✔

(B, F) → B wins. $\pi_{11}(B) = 0.5 \geq 1/3$

(C, E) → E wins. $\pi_{21}(E) = 5/9 \geq 1/3$

(B, E) → B needs to win w.p. ½, E needs to win w.p. ⅔

Feasibility of Reduced From (Cont'd)

- **A necessary condition** for *feasible single-item reduced form*:

  $\forall S_1 \subseteq T_1, \ldots, S_m \subseteq T_m,$

  Pr[$\exists$ i whose type is in S_i and gets the item] $\leq$ Pr[$\exists$ i whose type is in S_i]

- [Border '91, Border '07, Che-Kim-Mierendorff '11]:

  (*) is **also a sufficient condition** for feasibility.

BUT, too many subsets: need to check $2^{\sum_i |T_i|}$ conditions !!!

[C.-Daskalakis-Weinberg '

We can check feasibility

almost linear in $\Sigma_i|T_i|$,

i.e. the total number of bidd

type profiles).

Feasibility for Multi-item Reduced Form

Theorem [C.-Daskalakis-Weinberg '12]:
There is an poly-time algorithm that checks the feasibility of any multi-item reduced from.

❖ Remark:
- With this we can solve our succinct LP!
- The proof uses the ellipsoid method, separation Ξ optimization and sampling etc.
- Have many extensions, e.g. accommodates any combinatorial allocation constraints (unit-demand, single-minded...)

Implementation of a Feasible Reduced Form

- After solving the succinct LP, we find the optimal reduced form π^* and p^*.

- **Can you turn π^* and p^* into an auction whose reduced form is exactly π^* and p^*?**

- This is crucial, otherwise being able to solve the LP is meaningless.

- Will show you a way to implement any feasible reduced form, and it reveals important structure of the revenue-optimal auction!

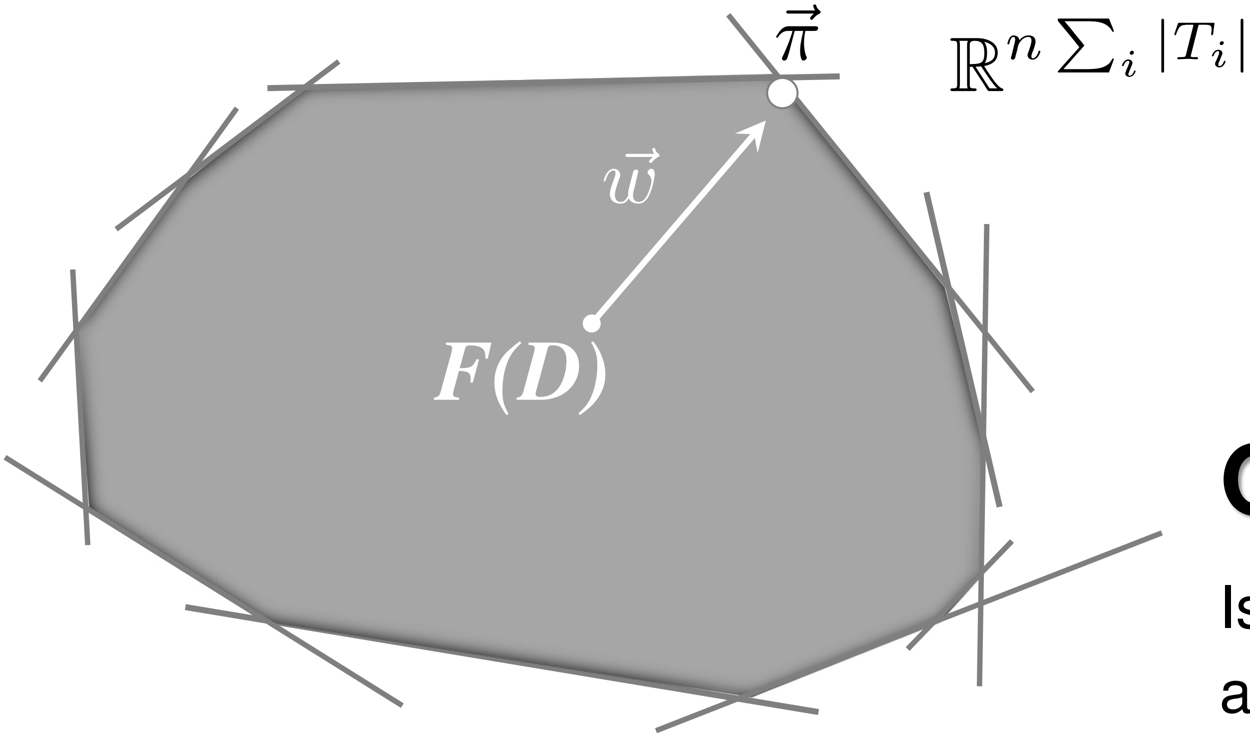
Implementation of a Feasible Reduced Form

Set of *Feasible* Reduced Forms

- Reduced form is collection $\{\pi_i : T_i \longrightarrow [0,1]^n\}$;
- Can view it as a vector $\vec{\pi} \in \mathbb{R}^{n \sum_i |T_i|}$;
- Let's call set of feasible reduced forms $F(D) \in \mathbb{R}^{n \sum_i |T_i|}$;

- **<u>Claim 1: *F(D)* is a convex polytope</u>**.
- **Proof:** ***Easy!***
  - A feasible reduced form $\vec{\pi}$ is implemented by a feasible allocation rule ***M***.
  - ***M*** is a distribution over deterministic feasible allocation rules, of which there is a finite number. So: $M = \sum_{\ell=1}^{k} p_\ell \cdot M_\ell$, where M_ℓ is **deterministic**.
  - Easy to see: $\vec{\pi} = \sum_{\ell=1}^{k} p_\ell \cdot \vec{\pi}(M_\ell)$
  - So, ***F(D)*** = $\left(\begin{array}{c}\text{convex hull of reduced forms of} \\ \text{feasible deterministic mechanisms}\end{array}\right)$

Set of *Feasible* Reduced Forms



Q:

Is there a **simple** allocation rule implementing the corners?