COMP/MATH 553 Algorithmic Game Theory
Lecture 9: Revenue Maximization in Multi-Dimensional Settings

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An overview of today's class

- *Myerson's Auction Recap*
- *Challenge of Multi-Dimensional Settings*
- *Unit-Demand Pricing*

Myerson's Auction Recap

[Myerson '81  **]** For any single-dimensional environment.

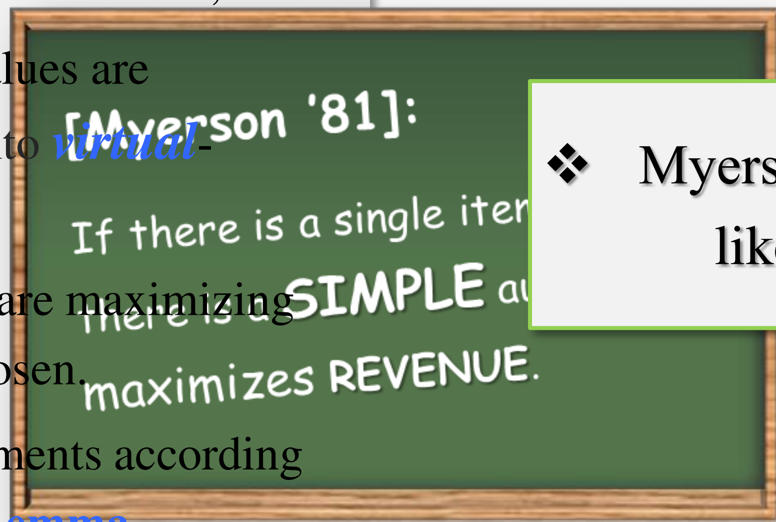
Let $F = F_1 \times F_2 \times \ldots \times F_n$ be the joint value distribution, and (x,p) be a DSIC mechanism. The expected revenue of this mechanism

$$E_{v\sim F}[\Sigma_i\ p_i(v)] = E_{v\sim F}[\Sigma_i\ x_i(v)\ \varphi_i\ (v_i)],$$

where $\varphi_i\ (v_i) := v_i - (1-F_i(v_i))/f_i(v_i)$ is called bidder i's virtual value (f_i is the density function for F_i).

Myerson’s Auction Recap

- Bidders report their values;
- The reported values are **transformed** into ***virtual***-values;
- the virtual-welfare maximizing allocation is chosen.
- Charge the payments according to **Myerson’s Lemma**.
- ***Transformation*** = depends on the distributions; ***deterministic*** function (the virtual value function);



❖ Myerson’s auction looks like the following

Nice Properties of Myerson's Auction

- **DSIC**, but optimal among all Bayesian Incentive Compatible (BIC) mechanisms!
- **Deterministic**, but optimal among all possibly randomized mechanisms!
- **Central open problem** in Mathematical Economics: How can we extend Myerson's result to **Multi-Dimensional Settings**?
- ***Important progress*** in the past a few years.
- See the ***Challenges*** first!

Challenges in Multi-Dimensional Settings

Example 1:

- A single buyer, 2 non-identical items
- Bidder is additive e.g. $v(\{1,2\}) = v_1+v_2$.
- Further simplify the setting, assume v_1 and v_2 are drawn i.i.d. from distribution F = U{1,2} (1 w.p. ½, and 2 w.p. ½).
- What's the optimal auction here?
- Natural attempt: How about sell both items using Myerson's auction separately?

Example 1:

- Selling each item separately with Myerson's auction has expected revenue \$2.
- Any other mechanism you might want to try?
- How about bundling the two items and offer it at \$3?
- What is the expected revenue?
- Revenue = $3 \times \Pr[v_1 + v_2 \geq 3] = 3 \times 3/4 = 9/4 > 2$!
- ***Lesson 1: Bundling Helps!!!***

Example 1:

- The effect of bundling becomes more **obvious** when the number of items is large.
- Since they are i.i.d., by the **central limit theorem** (or Chernoff bound) you know the bidder's value for the grand bundle (contains everything) will be a ***Gaussian distribution***.
- The ***variance*** of this distribution decreases ***quickly***.
- If set the price slightly lower than the expected value, then the bidder will buy the grand bundle w.p. almost 1. Thus, **revenue is almost the expected value!**
- This is the best you could hope for.

Example 2:

- Change ***F*** to be U{0,1,2}.
- Selling the items separately gives \$4/3.
- The best way to sell the Grand bundle is set it at price \$2, this again gives \$4/3.
- Any other way to sell the items?
- Consider the following menu. The bidder picks the best for her.
  - Buy either of the two items for \$2
  - Buy both for \$3

Example 2:

- Bidder's choice:

$v_1 \backslash v_2$	0	1	2
0	\$0	\$0	\$2
1	\$0	\$0	\$3
2	\$2	\$3	\$3

- Expected Revenue = $3 \times 3/9 + 2 \times 2/9 = 13/9 > 4/3$!

Example 3:

- Change $\boldsymbol{F_1}$ to be U{1,2}, $\boldsymbol{F_2}$ to be U{1,3}.

- Consider the following menu. The bidder picks the best for her.
  - Buy both items with price \$4.
  - ***A lottery***: get the first item for sure, and ***get the second item with prob. ½***. pay \$2.50.

- The expected revenue is \$2.65.

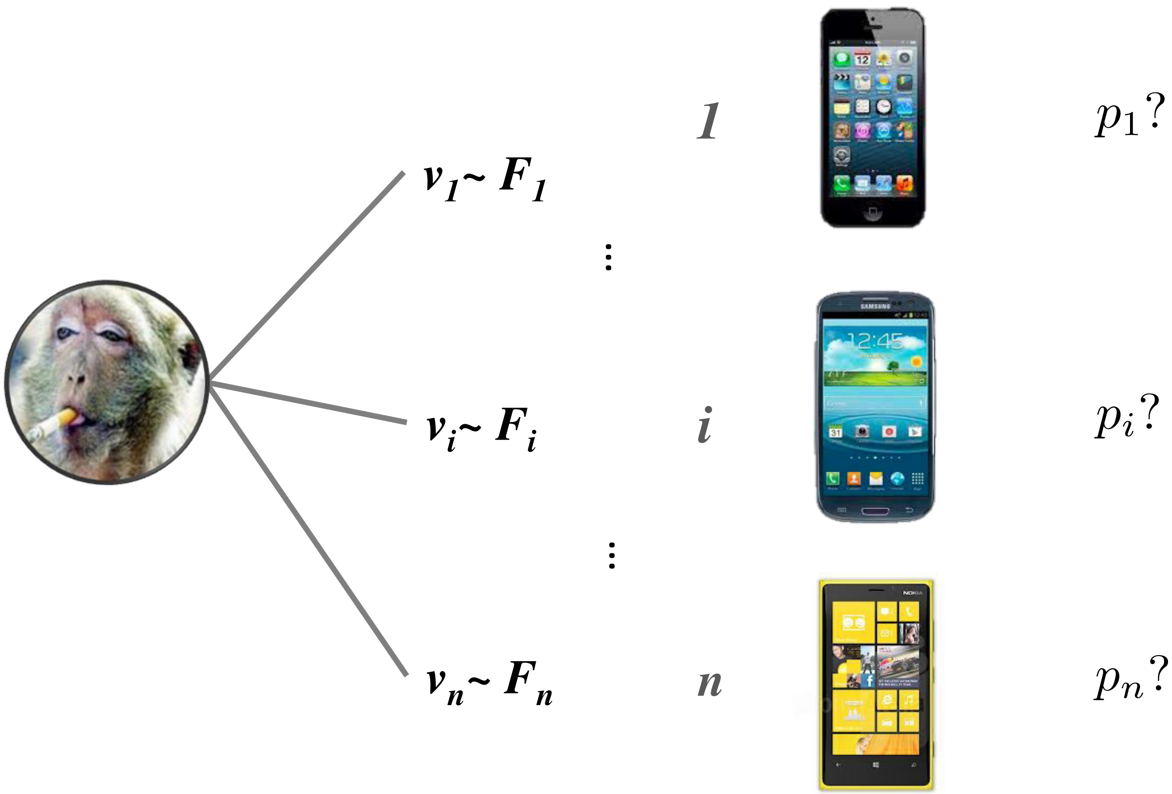
- Every deterministic auction — where every outcome awards either nothing, the first item, the second item, or both items — has **strictly less expected revenue**.

- ***Lesson 2: randomization could help!***

Unit-demand Bidder Pricing Problem

Unit-Demand Bidder Pricing Problem (UPP)

❑ A fundamental pricing problem



- Bidder chooses the item that maximizes $v_i - p_i$, if any of them is positive.
- Revenue will be the corresponding p_i.
- Focus on pricing only, not considering randomized ones.
- It's known randomized mechanism can only get a constant factor better than pricing.

Our goal for UPP

- Goal: design a pricing scheme that achieves a constant fraction of the revenue that is achievable by the optimal pricing scheme.

- Assumption: F_i's are regular.

Theorem [CHK '07]: There exists a simple pricing scheme (poly-time computable), that achieves at least **¼** of the revenue of the optimal pricing scheme.

Remark: the constant can be improved with a better analysis.

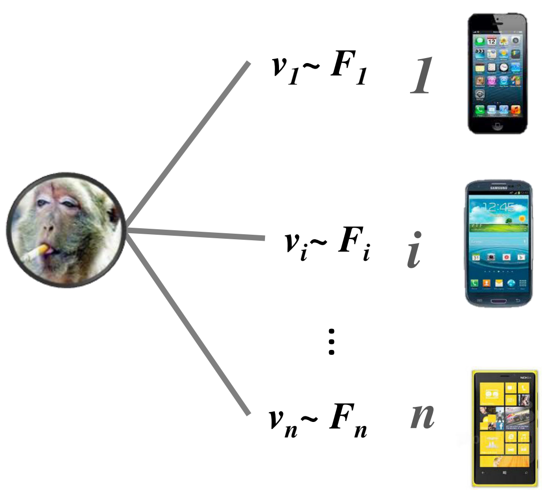
What is the Benchmark???

- When designing simple nearly-optimal auctions. The benchmark is clear.
- Myerson’s auction, or the miximum of the virtual welfare.
- In this setting we don’t know what the optimal pricing scheme looks like.
- We want to compare to the optimal revenue, but we have no clue what the optimal revenue is?
- Any natural upper bound for the optimal revenue?

Two Scenarios

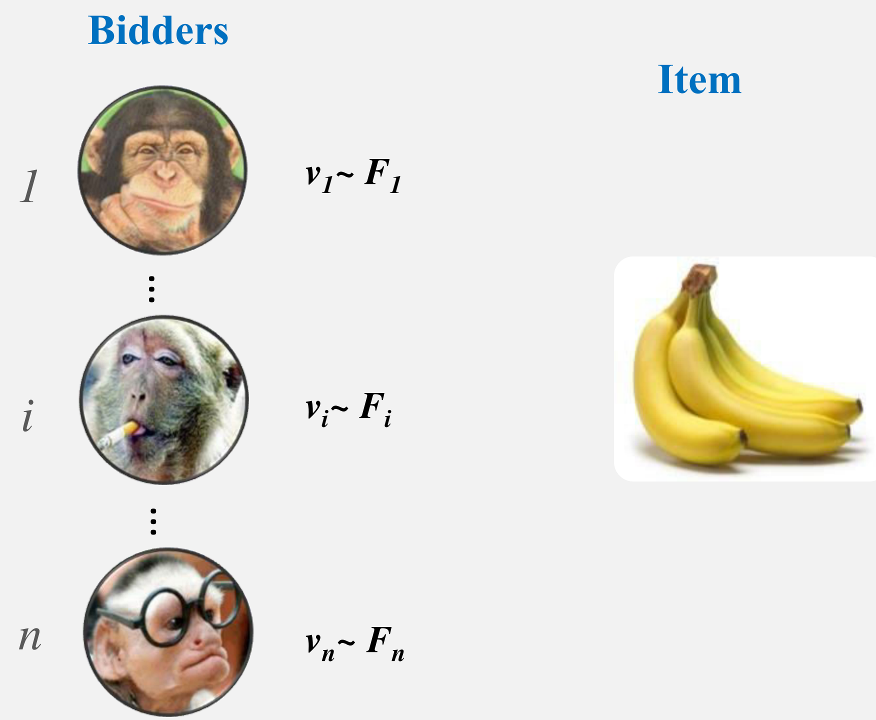
❖ (a) UPP

- ***One unit-demand bidder***
- ***n items***
- Bidder's value for the i-th item $\boldsymbol{v_i}$ is drawn independently from $\boldsymbol{F_i}$



❖ (b) Auction

- ***n bidders***
- ***One item***
- Bidder I's value for the item $\boldsymbol{v_i}$ is drawn independently from $\boldsymbol{F_i}$



Benchmark

Lemma 1: The optimal revenue achievable in scenario (a) is always less than the optimal revenue achievable in scenario (b).

- Proof: See the board.

- Remark: This gives a natural benchmark for the revenue in (a).

An even simpler benchmark

- In a single-item auction, the optimal expected revenue

$$\mathbf{E_{v\sim F}\,[max\ \Sigma_i\, \boldsymbol{x_i(v)\ \varphi_i\ (v_i)}] = E_{v\sim F}\,[\boldsymbol{max_i\, \varphi_i(v_i)^+}]} \text{ (the expected prize of the prophet)}$$

- Remember the following mechanism **RM** we learned in Lecture 6.
  1. Choose t such that $\mathbf{Pr[max_i}\ \varphi_i\ (v_i)^+ \geq t\mathbf{] = ½}$.
  2. Set a reserve price $r_i = \varphi_i^{-1}\ (t)$ for each bidder i with the t defined above.
  3. Give the item to the highest bidder that meets her reserve price (if any).
  4. Charge the payments according to Myerson's Lemma.

- By prophet inequality:

$$\mathbf{ARev(RM) = E_{v\sim F}\,[\Sigma_i\, \boldsymbol{x_i(v)\ \varphi_i\ (v_i)}] \geq ½\ E_{v\sim F}\,[\boldsymbol{max_i\, \varphi_i(v_i)^+}] = ½\ \boldsymbol{ARev(Myerson)}}$$

- Let's use the revenue of RM as the benchmark.

Inherent loss of this approach

- Relaxing the benchmark to be Myerson's revenue in (b)
- This step might lose a constant factor already.
- To get real optimal, a different approach is needed.