COMP/MATH 553 Algorithmic Game Theory
Lecture 7: Bulow-Klemperer & VCG Mechanism

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Yang Cai

An overview of today's class

- *Prior-Independent Auctions & Bulow-Klemperer Theorem*
- *General Mechanism Design Problems*
- *Vickrey-Clarke-Groves Mechanism*

Prior-Independent Auctions

Another Critique to the Optimal Auction

- What if your distributions are ***unknown***?
- When there are many bidders and enough past data, it is reasonable to assume you know exactly the value distributions.
- But if the market is “thin”, you might not be confident or not even know the value distributions.
- Can you design an auction that does not use any knowledge about the distributions but performs ***almost as well as*** if you know ***everything*** about the distributions.
- Active research agenda, called prior-independent auctions.

Bulow-Klemperer Theorem

[Bulow-Klemperer '96] For any regular distribution F and integer n.

$$\mathbb{E}_{v_1,\ldots,v_{n+1}}[\mathrm{REV}(\mathrm{Vickrey})] \geq \mathbb{E}_{v_1,\ldots,v_n}[\mathrm{REV}(\mathrm{Myerson})]$$

Remark:

- Vickrey's auction is prior-independent!
- This means with the same number of bidders, Vickrey Auction achieves at least n-1/n fraction of the optimal revenue. (exercise)
- More competition is better than finding the right auction format.

Proof of Bulow-Klemperer

- Consider another auction ***M*** with ***n+1*** bidders:
  1. Run Myerson on the first n bidders.
  2. If the item is unallocated, give it to the last bidder for free.

- This is a ***DSIC*** mechanism. It has the ***same*** revenue as Myreson's auction with n bidders.

- Notice that it's allocation rule always gives out the item.

- Vickrey Auction also always gives out the item, but always to the bidder who has the highest value (also with the highest virtual value).

- Vickrey Auction has the highest virtual welfare among all DSIC mechanisms that always give out the item! □

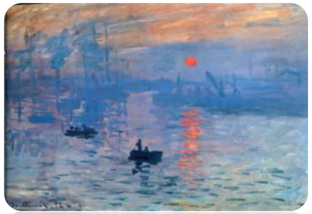
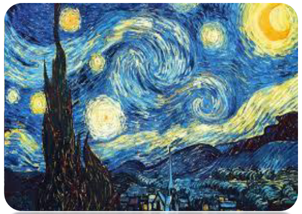
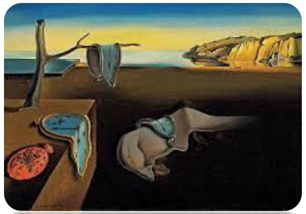
General Mechanism Design Problem (Multi-Dimensional)

Multi-Dimensional Environment

- So far, we have focused on single-dimensional environment.

- In many scenarios, bidders have different value for different items.

  - Sotherby's Auction:

- Multi-Dimensional Environment
  - $\boldsymbol{n}$ strategic participants/agents,
  - a set of possible outcomes $\boldsymbol{\Omega}$.
  - agent $\boldsymbol{i}$ has a private value $\boldsymbol{v_i(\omega)}$ for each $\boldsymbol{\omega}$ in $\boldsymbol{\Omega}$ (abstract and could be large).

Examples of Multi-Dimensional Environment

- Single-item Auction in the single-dimensional setting:
  - $n+1$ outcomes in Ω.
  - Bidder i only has positive value for the outcome in which he wins, and has value 0 for the rest n outcomes

- Single-item Auction in the multi-dimensional setting:
  - Imagine you are not selling an item, but auctioning a startup who has a lot of valuable patents.
  - n companies are competing for it.
  - Still $n+1$ outcomes in Ω.
  - But company i doesn't win, it might prefer the winner to be someone in a different market other than a direct competitor.
  - So besides the outcome that i wins, i has different values for the rest n outcomes.

How do you optimize Social Welfare (Non-bayesian)?

- What do I mean by optimize social welfare (algorithmically)?
  - $\boldsymbol{\omega^* := \textit{arg}\mathbf{max}_\omega \Sigma_i\, v_i(\omega)}$

- How do you design a DSIC mechanism that optimizes social welfare.
  - Take the same two-step approach.
  - Sealed-bid auction. Bidder $\boldsymbol{i}$ submits $\boldsymbol{b_i}$ which is indexed by Ω.
  - Allocation rule is clear: assume $\boldsymbol{b_i}$'s are the true values and choose the outcome that maximizes social welfare.
  - In single-dimensional settings, once the allocation rule is decided, Myerson's lemma tells us the unique payment rule.
  - In multi-dimensional settings, Myerson's lemma doesn't apply ... How can you define monotone allocation rule when bids are multi-dimensional?
  - Similarly, how can we define the payment rule even if we know the allocation rule.

Vickrey-Clarke-Groves (VCG) Mechanism

The VCG Mechanism

[The Vickrey-Clarke-Groves (VCG) Mechanism] In every general mechanism design environment, there is a DSIC mechanism that maximizes the social welfare. In particular the allocation rule is

$$x(b) = \mathrm{argmax}_{\omega} \ \Sigma_i \ b_i(\omega) \qquad (1);$$

and the payment rule is

$$p_i(b) = \max_{\omega} \ \Sigma_{j \neq i} \ b_j(\omega) - \Sigma_{j \neq i} \ b_j(\omega^*) \qquad (2),$$

where $\omega^* = \mathrm{argmax}_{\omega} \ \Sigma_i \ b_i(\omega)$ is the outcome chosen in (1).

Understand the payment rule

- What does the payment rule mean?
  - $p_i(b) = max_\omega\, \Sigma_{j\neq i}\, b_j(\omega) - \Sigma_{j\neq i} b_j(\omega^*)$
  - $max_\omega\, \Sigma_{j\neq i}\, b_j(\omega)$ is the optimal social welfare when i is not there.
  - ω^* is the optimal social welfare outcome, and $\Sigma_{j\neq i} b_j(\omega^*)$ is the welfare from all agents except i.
  - So the difference $max_\omega\, \Sigma_{j\neq i}\, b_j(\omega) - \Sigma_{j\neq i} b_j(\omega^*)$ can be viewed as "the ***welfare loss*** inflicted on the other $n-1$ agents by i's presence". Called "externality" in Economics.
  - Example: single-item auction.
    - If i is the winner, $max_\omega\, \Sigma_{j\neq i}\, b_j(\omega)$ is the second largest bid.
    - $\Sigma_{j\neq i} b_j(\omega^*) = 0.$
    - So exactly second-price.

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Proof: See the board!

Discussion of the VCG mechanism

- *DSIC* mechanism that *optimizes social welfare* for *any* mechanism design problem !

- However, sometimes ***impractical***.

- How do you find the allocation that maximizes social welfare. If Ω is really large, what do you do?
  - m items, n bidders, each bidder wants only one item.
  - m items, n bidders, each bidder is single-minded (only like a particular set of items).
  - m items, n bidders, each bidder can take any set of items.

- Computational intractable.

- If you use approximation alg., the mechanism is no longer DSIC.