Bidirectional Decision Procedures for the Intuitionistic Propositional Modal Logic IS4

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Abstract. We present a multi-context sequent calculus whose derivations are in bijective correspondence with normal natural deductions in the propositional fragment of the intuitionistic modal logic **IS4**. This calculus, suitable for proof enumeration, is the starting point for the development of a sequent calculus-based bidirectional decision procedure for propositional **IS4**. In this system, big-step inference rules are constructed in a forward direction, but searched over in a backward direction. We also present a variant which searches directly over normal natural deductions. Experimental results show that on most problems, the bidirectional prover is competitive with conventional backward provers using loop-detection and inverse method provers, significantly outperforming them in a number of cases.

1 Introduction

Intuitionistic modal logics are constructive logics incorporating operators of necessity ($\Box$) and possibility ($\Diamond$). Fitch [8], Prawitz [16], Satre [18], and more recently Simpson [19], Bierman and de Paiva [2], and Pfenning and Davies [15] have investigated a broad range of proof-theoretical properties of various logics of this kind. Recently, such logics have also found applications in hardware verification [7] and proposed type systems for staged computation [4] and distributed computing [13]. A logic frequently used in these settings is either the intuitionistic variant of the classical modal logic **S4**, which we will call **IS4**, or a logic that can be expressed through **IS4**, such as Fairtlough and Mendler's lax logic [7] (see for example [15] for the relationship between **IS4** and lax logic).

In this light, it is surprising that proof search in **IS4** has not received more attention. Howe has investigated proof enumeration and theorem proving in lax logic [12] and, coming closer to our work, has presented a backward decision procedure for the fragment of propositional **IS4** without the possibility modality [11]. His system performs loop-detection using a history mechanism, but is encumbered by a large number of rules and related provisos (21 axioms and inference rules). It would only grow with the addition of the possibility modality, which would also require a different loop-detection mechanism.

Our contributions begin with a sequent calculus for propositional **IS4** suitable for proof enumeration. This forms the basis for the development of a sequent calculus-based *bidirectional* **IS4** decision procedure, in which big-step inference rules are constructed in a forward direction, while derivations constructed from these rules are searched over in a backward direction. We also demonstrate that this approach corresponds very closely to an elegant bidirectional decision procedure that searches directly over normal natural deductions. The key to our theoretical justification of both of these decision procedures is a refinement of the well-known subformula property, which we use to restrict nondeterminism in focused proof search in the presence of multiple contexts. Although we concentrate on propositional **IS4**, we believe that the techniques presented are general enough to find applications in other constructive logics, such as contextual modal logic [14]. To evaluate our approach empirically, we have put together a set of 50 benchmark formulas for **IS4**. Experimental results show that on most problems, the bidirectional prover is competitive with conventional backward provers using loop-detection and inverse method provers, significantly outperforming them in a number of cases.

In Sect. 2 we summarize the relevant background and introduce our core natural deduction formalism, while Sect. 3 presents corresponding sequent calculi for proof search in both backward and forward directions. In Sect. 4 we discuss some of the intricacies of focused forward proof search, building up to the bidirectional decision procedures in Sects. 5 and 6. Experimental results are given in Sect. 7, while Sect. 8 concludes with related and future work.

2 Natural Deduction

Formulas in the propositional fragment of **IS4** are defined by

$$A \quad ::= \quad P \mid \bot \mid A_1 \supset A_2 \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \Box A \mid \Diamond A$$

where P is atomic and negation and truth are defined notationally in the usual way. Our starting point is a multi-context natural deduction formulation for **IS4** similar to that of Pfenning and Davies [15], except that only natural deductions in normal form can be constructed. This is achieved by annotating judgements with their intended direction of reasoning:

$\Delta; \Gamma \vdash A \uparrow$	A has a normal proof under hypotheses Δ and Γ,
$\Delta; \Gamma \vdash A \downarrow$	A can be extracted from hypotheses in Δ and Γ using only elimination rules,

where $\Gamma = A_1, \ldots, A_n$ is a context of true hypotheses and $\Delta = B_1, \ldots, B_m$ is a modal context of *valid* hypotheses whose truth does not depend on hypotheses about the truth of other formulas. The resulting calculus, which we will call $\mathbf{NJ}^{\mathbf{N}}_{\mathbf{IS4}}$, is shown in Fig. 1. The usual structural properties of weakening, contraction, and exchange hold for both contexts. Note that while $\mathbf{NJ}^{\mathbf{N}}_{\mathbf{IS4}}$ defines the normal forms that we are interested in during proof search, an unrestricted variant $\mathbf{NJ}_{\mathbf{IS4}}$ can be obtained by dropping the arrow annotations and the rule $\uparrow\downarrow$. Using an approach analogous to that of Howe in [10], the two systems can be shown to be equivalent in terms of provability.

$$\frac{}{\Delta; \Gamma, A, \Gamma' \vdash A \downarrow}\,\mathrm{hyp}_1 \qquad \frac{}{\Delta, A, \Delta'; \Gamma \vdash A \downarrow}\,\mathrm{hyp}_2 \qquad \frac{\Delta; \Gamma \vdash \bot \downarrow}{\Delta; \Gamma \vdash C \uparrow}\,\bot\mathrm{E}$$

$$\frac{\Delta; \Gamma, A_1 \vdash A_2 \uparrow}{\Delta; \Gamma \vdash A_1 \supset A_2 \uparrow}\,{\supset}\mathrm{I} \qquad \frac{\Delta; \Gamma \vdash A_1 \supset A_2 \downarrow \quad \Delta; \Gamma \vdash A_1 \uparrow}{\Delta; \Gamma \vdash A_2 \downarrow}\,{\supset}\mathrm{E}$$

$$\frac{\Delta; \Gamma \vdash A_1 \uparrow \quad \Delta; \Gamma \vdash A_2 \uparrow}{\Delta; \Gamma \vdash A_1 \wedge A_2 \uparrow}\,\wedge\mathrm{I} \qquad \frac{\Delta; \Gamma \vdash A_1 \wedge A_2 \downarrow}{\Delta; \Gamma \vdash A_j \downarrow}\,\wedge\mathrm{E}_j$$

$$\frac{\Delta; \Gamma \vdash A_j \uparrow}{\Delta; \Gamma \vdash A_1 \vee A_2 \uparrow}\,\vee\mathrm{I}_j \qquad \frac{\Delta; \Gamma \vdash A_1 \vee A_2 \downarrow \quad \Delta; \Gamma, A_1 \vdash C \uparrow \quad \Delta; \Gamma, A_2 \vdash C \uparrow}{\Delta; \Gamma \vdash C \uparrow}\,\vee\mathrm{E}$$

$$\frac{\Delta; \cdot \vdash A \uparrow}{\Delta; \Gamma \vdash \Box A \uparrow}\,\Box\mathrm{I} \qquad \frac{\Delta; \Gamma \vdash \Box A \downarrow \quad \Delta, A; \Gamma \vdash C \uparrow}{\Delta; \Gamma \vdash C \uparrow}\,\Box\mathrm{E}$$

$$\frac{\Delta; \Gamma \vdash A \uparrow}{\Delta; \Gamma \vdash \Diamond A \uparrow}\,\Diamond\mathrm{I} \qquad \frac{\Delta; \Gamma \vdash \Diamond A \downarrow \quad \Delta; A \vdash \Diamond C \uparrow}{\Delta; \Gamma \vdash \Diamond C \uparrow}\,\Diamond\mathrm{E} \qquad \frac{\Delta; \Gamma \vdash A \downarrow}{\Delta; \Gamma \vdash A \uparrow}\,{\uparrow\downarrow} \qquad j \in \{1, 2\}$$

Fig. 1. $\mathbf{NJ}^{\mathbf{N}}_{\mathbf{IS4}}$

The inference rules of $\mathbf{NJ}^{\mathbf{N}}_{\mathbf{IS4}}$ are largely standard, but to glean some intuition about the rules involving $\Box$ and $\Diamond$, it is useful to think of the modalities as quantifying truth over worlds in some universe. To say that $\Box A$ is true is to say that A is true in all worlds, while to say that $\Diamond A$ is true is to say that A is true in some world. Under this interpretation, the hypotheses in the modal context can be used in all worlds, while those in the regular context can only be used in the current world, in which we are trying to prove the succedent.

3 Sequent Calculi

Following the approach of Dyckhoff and Pinto [6], we can construct a *focused* sequent calculus for propositional **IS4** whose derivations are in bijective correspondence with normal natural deductions. This system, which we will call $\mathbf{MJ}_{\mathbf{IS4}}$, is shown in Fig. 2 and involves two forms of sequents:

$\Delta; \Gamma \to C$	C can be proved from assumptions Δ, Γ,
$\Delta; \Gamma \rhd A \to C$	C can be proved from assumptions Δ, Γ, A, focusing on the assumption A.

If a sequent is focused on a formula A, then the only applicable rules are those with A as a principal formula. Following Girard [9], we will call the position of the focused formula the *stoup*.

Theorem 1. *Derivations of unfocused sequents in* $\mathbf{MJ}_{\mathbf{IS4}}$ *correspond bijectively to derivations in* $\mathbf{NJ}^{\mathbf{N}}_{\mathbf{IS4}}$.

$$\frac{A \text{ is atomic}}{\Delta; \Gamma \triangleright A \to A}\,\text{init} \qquad \frac{}{\Delta; \Gamma \triangleright \bot \to C}\,\bot\text{L}$$

$$\frac{\Delta; \Gamma, A, \Gamma' \triangleright A \to C}{\Delta; \Gamma, A, \Gamma' \to C}\,\text{ch}_1 \qquad \frac{\Delta, A, \Delta'; \Gamma \triangleright A \to C}{\Delta, A, \Delta'; \Gamma \to C}\,\text{ch}_2$$

$$\frac{\Delta; \Gamma, A_1 \to A_2}{\Delta; \Gamma \to A_1 \supset A_2}\,\supset\text{R} \qquad \frac{\Delta; \Gamma \to A_1 \quad \Delta; \Gamma \triangleright A_2 \to C}{\Delta; \Gamma \triangleright A_1 \supset A_2 \to C}\,\supset\text{L}$$

$$\frac{\Delta; \Gamma \to A_1 \quad \Delta; \Gamma \to A_2}{\Delta; \Gamma \to A_1 \wedge A_2}\,\wedge\text{R} \qquad \frac{\Delta; \Gamma \triangleright A_j \to C}{\Delta; \Gamma \triangleright A_1 \wedge A_2 \to C}\,\wedge\text{L}_j$$

$$\frac{\Delta; \Gamma \to A_j}{\Delta; \Gamma \to A_1 \vee A_2}\,\vee\text{R}_j \qquad \frac{\Delta; \Gamma, A_1 \to C \quad \Delta; \Gamma, A_2 \to C}{\Delta; \Gamma \triangleright A_1 \vee A_2 \to C}\,\vee\text{L}$$

$$\frac{\Delta; \cdot \to A}{\Delta; \Gamma \to \Box A}\,\Box\text{R} \qquad \frac{\Delta, A; \Gamma \to C}{\Delta; \Gamma \triangleright \Box A \to C}\,\Box\text{L}$$

$$\frac{\Delta; \Gamma \to A}{\Delta; \Gamma \to \Diamond A}\,\Diamond\text{R} \qquad \frac{\Delta; A \to \Diamond C}{\Delta; \Gamma \triangleright \Diamond A \to \Diamond C}\,\Diamond\text{L} \qquad j \in \{1, 2\}$$

Fig. 2. $\mathbf{MJ_{IS4}}$

Proof. Injective functions can be constructed, mapping derivations from $\mathbf{NJ^N_{IS4}}$ to $\mathbf{MJ_{IS4}}$ and back. This is a straightforward extension of the approach used by Dyckhoff and Pinto for regular intuitionistic propositional logic in [6]. $\square$

Although $\mathbf{MJ_{IS4}}$ is suitable for proof search in a backward direction, a naive approach still requires loop-detection to achieve a decision procedure. While this is possible, we will not pursue this direction further here, but instead concentrate on forward proof search, and on how we can combine ideas from backward and forward proof search to achieve bidirectional decision procedures.

Constructing $\mathbf{MJ_{IS4}}$ proofs from the top down is complicated by the presence of multiple contexts, making it less than ideal for forward proof search. All $\mathbf{MJ_{IS4}}$ derivations begin, at the leaves, with focused sequents of the form $\Delta; \Gamma \triangleright A \to A$, with A atomic. After a sequence of (possibly zero) left-rule applications, the stoup formula is dropped from the stoup into one of the contexts by an application of ch_1 or ch_2. In a focused forward calculus used as the basis for the inverse method [5], we would proceed in a similar way, but it is not clear which context a stoup formula should be dropped into.

To address this uncertainty, we refine the idea of focusing and develop the system $\mathbf{MJ^F_{IS4}}$, which is suitable for forward proof search and features sequents of three kinds, involving both modal and nonmodal stoups:

$\Delta; \Gamma \mapsto C$	C can be proved using all assumptions in Δ, Γ,
$\Delta; \Gamma \triangleright A \mapsto C$	C can be proved using all assumptions in Δ, Γ, A, with A assumed true,

$$\frac{A \text{ is atomic}}{\cdot;\cdot \rhd^i A \mapsto A}\text{init}_i \qquad \frac{}{\cdot;\cdot \rhd^i \bot \mapsto C}\bot\text{L}_i \qquad \frac{\Delta;\Gamma \rhd A \mapsto C}{\Delta;\Gamma,A \mapsto C}\text{ch}_1 \qquad \frac{\Delta;\Gamma \rhd\rhd A \mapsto C}{\Delta,A;\Gamma \mapsto C}\text{ch}_2$$

$$\frac{\Delta;\Gamma,A_1 \mapsto A_2}{\Delta;\Gamma \mapsto A_1 \supset A_2}\supset\text{R}_1 \qquad \frac{\Delta;\Gamma \mapsto A_2}{\Delta;\Gamma \mapsto A_1 \supset A_2}\supset\text{R}_2 \qquad \frac{\Delta_1;\Gamma_1 \mapsto A_1 \quad \Delta_2;\Gamma_2 \rhd^i A_2 \mapsto C}{\Delta_1,\Delta_2;\Gamma_1,\Gamma_2 \rhd^i A_1 \supset A_2 \mapsto C}\supset\text{L}_i$$

$$\frac{\Delta_1;\Gamma_1 \mapsto A_1 \quad \Delta_2;\Gamma_2 \mapsto A_2}{\Delta_1,\Delta_2;\Gamma_1,\Gamma_2 \mapsto A_1 \wedge A_2}\wedge\text{R} \qquad \frac{\Delta;\Gamma \rhd^i A_j \mapsto C}{\Delta;\Gamma \rhd^i A_1 \wedge A_2 \mapsto C}\wedge\text{L}_{i,j}$$

$$\frac{\Delta;\Gamma \mapsto A_j}{\Delta;\Gamma \mapsto A_1 \vee A_2}\vee\text{R}_j \qquad \frac{\Delta_1;\Gamma_1,A_1 \mapsto C \quad \Delta_2;\Gamma_2,A_2 \mapsto C}{\Delta_1,\Delta_2;\Gamma_1,\Gamma_2 \rhd^i A_1 \vee A_2 \mapsto C}\vee\text{L}_i$$

$$\frac{\Delta;\cdot \mapsto A}{\Delta;\cdot \mapsto \Box A}\Box\text{R} \qquad \frac{\Delta,A;\Gamma \mapsto C}{\Delta;\Gamma \rhd^i \Box A \mapsto C}\Box\text{L}_i \qquad \frac{\Delta;\Gamma \mapsto A}{\Delta;\Gamma \mapsto \Diamond A}\Diamond\text{R} \qquad \frac{\Delta;A \mapsto \Diamond C}{\Delta;\cdot \rhd^i \Diamond A \mapsto \Diamond C}\Diamond\text{L}_i$$

$$i,j \in \{1,2\}$$

Fig. 3. $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$

$\Delta;\Gamma \rhd\rhd A \mapsto C$	C can be proved using all assumptions in Δ,Γ,A, with A assumed valid.

Note that the forms of the focused sequents reveal which context the stoup formula will drop into. For brevity, we write $\Delta;\Gamma \rhd^i A \mapsto C$, $i \in \{1,2\}$ for either form of focused sequent.

The inference rules of $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$, shown in Fig. 3, are obtained by reinterpreting the rules of $\mathbf{MJ}_{\mathbf{IS4}}$ in a forward fashion and by defining the ch rules to behave as sketched above. The contexts of $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$, however, are interpreted very differently, in that sequents $\Delta;\Gamma \mapsto C$ and $\Delta;\Gamma \rhd^i A \mapsto C$, $i \in \{1,2\}$ assert that all assumptions in Δ and Γ, as well as A if the sequent is focused, are needed to prove C. General weakening, which holds in $\mathbf{MJ}_{\mathbf{IS4}}$, is thus disallowed, but local weakening is incorporated in the rule $\supset\text{R}_2$. We will think of contexts in $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ as sets and write Γ_1,Γ_2 and Γ,A for $\Gamma_1 \cup \Gamma_2$ and $\Gamma \cup \{A\}$, respectively.

Theorem 2. $\mathbf{MJ}_{\mathbf{IS4}}$ *and* $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ *are equivalent in terms of provability:*

1. *(a) If $\Delta;\Gamma \mapsto C$, then $\Delta;\Gamma \to C$, and*
   (b) if $\Delta;\Gamma \rhd^i A \mapsto C, i \in \{1,2\}$, then $\Delta;\Gamma \rhd A \to C$.
2. *(a) If $\Delta;\Gamma \to C$, then $\Delta';\Gamma' \mapsto C$, where $\Delta' \subseteq \Delta$, $\Gamma' \subseteq \Gamma$,*
   (b) if $\Delta;\Gamma \rhd A \to C$ and A is a subformula of a formula in Γ, then either $\Delta';\Gamma' \mapsto C$ or $\Delta';\Gamma' \rhd A \mapsto C$, where $\Delta' \subseteq \Delta$, $\Gamma' \subseteq \Gamma$, and
   (c) if $\Delta;\Gamma \rhd A \to C$ and A is a subformula of a formula in Δ, then either $\Delta';\Gamma' \mapsto C$ or $\Delta';\Gamma' \rhd\rhd A \mapsto C$, where $\Delta' \subseteq \Delta$, $\Gamma' \subseteq \Gamma$.

Proof. In both cases by simultaneous induction on the structure of the given derivation, using weakening in $\mathbf{MJ}_{\mathbf{IS4}}$ where necessary. □

Before turning to the bidirectional decision procedures that can be developed from $\mathbf{MJ_{IS4}}$ and $\mathbf{MJ^F_{IS4}}$, it is worthwhile to consider focused forward proof search by itself.

4 Focused Forward Proof Search

The forward calculus $\mathbf{MJ^F_{IS4}}$ suggests itself as a basis for an implementation of the inverse method [5], fundamental to which is the classification of the subformulas of a query formula into positive and negative classes. The sign of a subformula determines where in a sequent it may occur (for instance, as a goal formula or in the context) and restricts nondeterminism during proof search. We will refine this notion by classifying subformulas as either

1. positive ($+$) subformulas, which may occur as goal formulas,
2. negative ($-$) subformulas, which may occur in the nonmodal context,
3. negative focused ($\sim$) subformulas, which may occur in the nonmodal stoup,
4. valid ($=$) subformulas, which may occur in the modal context, or
5. valid focused ($\approx$) subformulas, which may occur in the modal stoup.

With this intended interpretation, it is straightforward to read the formal definition of refined signed subformulas directly from the inference rules of $\mathbf{MJ^F_{IS4}}$.

Definition 1. *A* signed subformula A^* *is a formula A with a sign* $* \in \{+, -, \sim, =, \approx\}$. *The subformula relation $\leq$ is the smallest reflexive and transitive relation between signed subformulas satisfying the following.*

$$A_1^-, A_2^+ \leq (A_1 \supset A_2)^+ \quad A_i^+ \leq (A_1 \wedge A_2)^+ \quad A_i^+ \leq (A_1 \vee A_2)^+$$
$$A^+ \leq (\Box A)^+ \quad A^+ \leq (\Diamond A)^+ \quad A^\sim \leq A^-$$
$$A_1^+, A_2^\sim \leq (A_1 \supset A_2)^\sim \quad A_i^\sim \leq (A_1 \wedge A_2)^\sim \quad A_i^- \leq (A_1 \vee A_2)^\sim$$
$$A^= \leq (\Box A)^\sim \quad A^- \leq (\Diamond A)^\sim \quad A^\approx \leq A^=$$
$$A_1^+, A_2^\approx \leq (A_1 \supset A_2)^\approx \quad A_i^\approx \leq (A_1 \wedge A_2)^\approx \quad A_i^- \leq (A_1 \vee A_2)^\approx$$
$$A^= \leq (\Box A)^\approx \quad A^- \leq (\Diamond A)^\approx \quad i \in \{1, 2\}$$

Note that for every negative subformula A^- of a signed formula C^*, C^* also has, as a subformula, the corresponding negative focused subformula $A^\sim$. The converse, however, is not true in general. A similar relation holds for valid and valid focused subformulas. Also, the usual signed subformula property extends to encompass our refined signing scheme, where we write Γ^- and $\Delta^=$ for contexts of signed subformulas of the forms $A_1^-, \ldots, A_n^-$ and $B_1^=, \ldots, B_m^=$, respectively.

Theorem 3. *Every sequent in an* $\mathbf{MJ^F_{IS4}}$ *derivation of*

$$\Delta^=; \Gamma^- \mapsto C^+ \quad \textit{or} \quad \Delta^=; \Gamma^- \rhd^i A^* \mapsto C^+, i \in \{1, 2\},$$

where $*$ is $\sim$ or $\approx$ if $i = 1$ or $i = 2$, respectively, is of the form

1. $D_1^=, \dots, D_n^=; E_1^-, \dots, E_m^- \mapsto F^+$,
2. $D_1^=, \dots, D_n^=; E_1^-, \dots, E_m^- \rhd E^\sim \mapsto F^+$, *or*
3. $D_1^=, \dots, D_n^=; E_1^-, \dots, E_m^- \rhd\rhd D^\approx \mapsto F^+$,

where all $D_j^=$, E_k^-, and $E^\sim$, $D^\approx$, and F^+ are signed subformulas of $\Delta^=$, Γ^-, C^+, and A^*.

Proof. By simultaneous induction on the structure of the given derivation. □

Theorem 3 guarantees, for instance, that in any $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ derivation of the sequent $\Delta^=; \Gamma^- \mapsto C^+$, all leaves are of the forms

$$\frac{A \text{ is atomic}}{\cdot; \cdot \rhd^i A^* \mapsto A^+}\,\mathrm{init}_i \qquad \text{or} \qquad \frac{}{\cdot; \cdot \rhd^i \bot^* \mapsto B^+}\,\bot\mathrm{L}_i \qquad i \in \{1, 2\}$$

where $*$ is $\sim$ or $\approx$ if $i = 1$ or $i = 2$, respectively, and A^*, A^+, $\bot^*$ and B^+ must be signed subformulas of $\Delta^=$, Γ^- and C^+. In general, every rule application considered by an implementation of the inverse method must abide by the conditions set forth by the extended signed subformula property. This provides a foundation for a focused inverse method prover for **IS4**, with nondeterminism restricted more strongly than by the usual subformula property. However, this idea becomes even more interesting when combined with some backward proof search ideas, discussed next.

5 Bidirectional Sequent Calculus Method

The idea behind the bidirectional sequent calculus method is that given a query formula A, we can, by exploiting forward proof search, construct a set of derived big-step inference rules for $\mathbf{MJ}_{\mathbf{IS4}}$ which conceal all left-rule applications that could be needed in a proof of A. We then carry out backward proof search over these big-step rules and the usual right-rules of $\mathbf{MJ}_{\mathbf{IS4}}$. By design, our big-step inference rules will correspond exactly to the notion of *focused threads* in $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ derivations, defined as follows.

Definition 2. *A* focused thread *of an $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ derivation is a segment of the derivation that begins, at the top, with an application of $init_i$, $\vee L_i$, $\Box L_i$, or $\Diamond L_i$, $i \in \{1, 2\}$ (raising a formula into either stoup in the conclusion), includes only focused sequents, and ends with an application of ch_i, $i \in \{1, 2\}$ (dropping a formula from the stoup).*

In any $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ derivation, left-rule applications necessarily occur in focused threads, so we can think of derivations as consisting of focused threads strung together using right-rule applications. The key insight is that all focused threads that might be needed in an $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ proof of a formula A can be nondeterministically constructed prior to proof search by inspecting the structure and the subformulas of A. To justify this claim, we will use our refined subformula property.

First note that it is straightforward to uniquely label subformula occurrences of a formula to be proved, and that the definition of signed subformulas, the signed subformula property, and the inference rules of $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ can be adjusted to operate on labels rather than formulas, thus differentiating between subformula occurrences. Moreover, by inspecting the inference rules of $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$, every proof of a labelled formula has the property that for every sequent occurring in the proof, no labels are duplicated in the contexts and goal formula. The stoup formula, however, may have the same label as some subformula in one of the contexts.

To give some intuition as to how to construct all the focused threads possibly needed for a proof of a formula, we will illustrate the approach on the following small example:

$$\overbrace{\overbrace{\Box(\overbrace{\overbrace{A}^{L_3^+} \supset \overbrace{B}^{L_4^{\approx}}}^{L_2^{=},L_2^{\approx}})}^{L_1^-,L_1^{\sim}} \supset \overbrace{\Diamond(\overbrace{\overbrace{A}^{L_7^-,L_7^{\sim}} \supset \overbrace{B}^{L_8^+}}^{L_6^+})}^{L_5^+}}^{L_0^+}$$

with subformulas

$$L_0^+, L_3^+, L_5^+, L_6^+, L_8^+, \qquad L_1^-, L_7^-, \qquad L_1^{\sim}, L_7^{\sim}, \qquad L_2^{=}, \qquad \text{and} \qquad L_2^{\approx}, L_4^{\approx}.$$

The signed subformula property guarantees that in a proof of the sequent $\cdot;\cdot \mapsto L_0^+$, the only axioms we require are

$$\frac{}{\cdot;\cdot \rhd L_7^{\sim} \mapsto L_3^+}\,\mathrm{init}_1 \qquad \text{and} \qquad \frac{}{\cdot;\cdot \rhd \rhd L_4^{\approx} \mapsto L_8^+}\,\mathrm{init}_2$$

Consider the first of these axioms. Every left-rule either drops the stoup formula into a context or expands it. The immediate parent of $L_7^{\sim}$ in the subformula hierarchy is L_7^-, indicating that dropping L_7 into the context is a permissible operation. In fact, it is the *only* operation permitted by the signed subformula property operating on labels. We can collapse this short focused thread into a single derived big-step inference rule:

$$\frac{\dfrac{}{\cdot;\cdot \rhd L_7^{\sim} \mapsto L_3^+}\,\mathrm{init}_1}{\cdot; L_7^- \mapsto L_3^+}\,\mathrm{ch}_1 \qquad \rightsquigarrow \qquad \frac{}{\cdot; L_7^- \mapsto L_3^+}\,(1)$$

Considering the second axiom, we note that the parent subformula of $L_4^{\approx}$ is $L_2^{\approx}$, also a focused subformula. The next rule application should then be $\supset\mathrm{L}_2$, with $L_2^{\approx}$ as the principal formula. In fact, it is not difficult to see that since every subformula occurrence has a unique parent subformula, the signed subformula property operating on labels always uniquely dictates which rule may be applied. This game continues until the end of the focused thread. In the case of the second axiom, the immediate parent of $L_2^{\approx}$ is $L_2^{=}$, signalling an application of ch_2 and the end of the thread:

$$
\dfrac{\Delta; \Gamma \mapsto L_3^+ \qquad \dfrac{}{\cdot; \cdot \rhd\!\rhd L_4^{\approx} \mapsto L_8^+}\,\mathrm{init}_2}{\dfrac{\Delta; \Gamma \rhd\!\rhd L_2^{\approx} \mapsto L_8^+}{\Delta, L_2^{=}; \Gamma \mapsto L_8^+}\,\mathrm{ch}_2}\,{\supset}\mathrm{L}_2
\qquad \rightsquigarrow \qquad
\dfrac{\Delta; \Gamma \mapsto L_3^+}{\Delta, L_2^{=}; \Gamma \mapsto L_8^+}\,(2)
$$

Note that this thread, unlike the one concealed by (1), has open premises and is parametric in the contexts Δ and Γ. Finally, the signed subformula property allows one more focused thread, starting with

$$
\dfrac{\Delta, L_2^{=}; \Gamma \mapsto M^+}{\Delta; \Gamma \rhd L_1^{\sim} \mapsto M^+}\,\Box\mathrm{L}_1
$$

The immediate parent subformula of $L_1^{\sim}$ is L_1^-, so this thread ends here, yielding the big-step rule

$$
\dfrac{\dfrac{\Delta, L_2^{=}; \Gamma \mapsto M^+}{\Delta; \Gamma \rhd L_1^{\sim} \mapsto C^+}\,\Box\mathrm{L}_1}{\Delta; \Gamma, L_1^- \mapsto M^+}\,\mathrm{ch}_1
\qquad \rightsquigarrow \qquad
\dfrac{\Delta, L_2^{=}; \Gamma \mapsto M^+}{\Delta; \Gamma, L_1^- \mapsto M^+}\,(3)
$$

Notice that this big step rule is schematic not only in the contexts Δ and Γ, but also in the goal formula M^+. Since the signed subformula property allows no other focused threads, the remainder of the proof, if one exists, may only chain the derived big-step rules (1), (2) and (3) together with right-rule applications. In this case, completing the proof is straightforward:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot; L_7^- \mapsto L_3^+}\,(1)}{L_2^{=}; L_7^- \mapsto L_8^+}\,(2)}{L_2^{=}; \cdot \mapsto L_6^+}\,{\supset}\mathrm{R}}{L_2^{=}; \cdot \mapsto L_5^+}\,\Diamond\mathrm{R}}{\cdot; L_1^- \mapsto L_5^+}\,(3)}{\cdot; \cdot \mapsto L_0^+}\,{\supset}\mathrm{R}
$$

In general, to cover all focused threads, the construction of big-step rules must begin with focused sequents of the following kinds, where $*$ is $\sim$ or $\approx$, depending on whether $i = 1$ or $i = 2$:

1. $\cdot; \cdot \rhd^i L_j^* \mapsto L_k^+$, where L_j and L_k denote the same atomic formula,
2. $\cdot; \cdot \rhd^i L_j^* \mapsto M^+$, where L_j denotes $\bot$ and M is schematic,
3. $\Delta; \Gamma \rhd^i L_j^* \mapsto M^+$, where L_j denotes $A_1 \vee A_2$ and M is schematic,

4. $\Delta;\Gamma \triangleright^i L_j^* \mapsto M^+$, where L_j denotes $\Box A$ and M is schematic, and
5. $\Delta;\Gamma \triangleright^i L_j^* \mapsto M^+$, where L_j denotes $\Diamond A$ and M denotes $\Diamond C$, C being schematic.

Moreover, the constructed big-step rules must end with a stoup formula dropped into one of the contexts.

The question now is how these forward-constructed big-step rules can complement backward proof search. The key observation is that every focused thread of an $\mathbf{MJ^F_{IS4}}$ derivation can be converted into a focused thread of an $\mathbf{MJ_{IS4}}$ derivation simply by applying weakening, reducing valid focused sequents to focused sequents and omitting the now unecessary signs of subformula labels. For instance,

$$\dfrac{\dfrac{\Delta;\Gamma \mapsto L_3^+ \qquad \dfrac{}{\cdot;\cdot \triangleright \triangleright L_4^{\approx} \mapsto L_8^+}\,\mathrm{init}_2}{\Delta;\Gamma \triangleright \triangleright L_2^{\approx} \mapsto L_8^+}\,{\supset}\mathrm{L}_2}{\Delta, L_2^{=};\Gamma \mapsto L_8^+}\,\mathrm{ch}_2$$

can be converted into

$$\dfrac{\dfrac{\Delta, L_2;\Gamma \to L_3 \qquad \dfrac{}{\Delta, L_2;\Gamma \triangleright L_4 \to L_8}\,\mathrm{init}}{\Delta, L_2;\Gamma \triangleright L_2 \to L_8}\,{\supset}\mathrm{L}}{\Delta, L_2;\Gamma \to L_8}\,\mathrm{ch}_2$$

This makes it possible to construct big-step rules for $\mathbf{MJ_{IS4}}$. The benefit of performing backward proof search over these big-step rules and the remaining right rules is that it requires no loop-detection, with one requirement: that every big-step rule is used at most once along every branch of the proof, from root to leaf. The following result of $\mathbf{MJ^F_{IS4}}$ guarantees that this requirement does not cost us completeness.

Theorem 4. *Every derivation of a sequent $\Delta^{=};\Gamma^{-} \mapsto M^+$ or $\Delta^{=};\Gamma^{-} \triangleright^i L^* \mapsto M^+$, $i \in \{1,2\}$, has the property that no branch (from root to leaf) contains two focused sequents $\Delta_1^{=};\Gamma_1^{-} \triangleright^j K^* \mapsto M_1^+$ and $\Delta_2^{=};\Gamma_2^{-} \triangleright^j K^* \mapsto M_2^+$, $j \in \{1,2\}$ with the same stoup formula occurrence (K^*).*

Proof. Assuming that a stoup formula occurrence is repeated along a branch can be shown to contradict the fact that no subformula occurrence may be duplicated in the contexts. $\Box$

Since the identity of a focused thread depends on the identities of the focused formulas it contains, and since focused threads contain at least one focused sequent, we obtain the following important corollary.

Corollary 1. *Every derivation of a sequent $\Delta;\Gamma \mapsto M$ or $\Delta;\Gamma \triangleright^i L \mapsto M$, $i \in \{1,2\}$ has the property that no focused thread occurs twice along a branch.*

The consequence of this result is that if a sequent is provable in $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$, it is provable without using any big-step rule twice along a branch. This means that there exists a corresponding proof in $\mathbf{MJ}_{\mathbf{IS4}}$ that also uses every big-step rule at most once along every branch. With the observation that every right rule of $\mathbf{MJ}_{\mathbf{IS4}}$ reduces the complexity of the goal formula, this means that every rule application during proof search either reduces the number of available big-step rules along the current branch, or leaves the number of available big-step rules unmodified but reduces the complexity of the goal formula. This measure gives an immediate termination guarantee without the need for loop-detection. All that is needed is a way of keeping track of which big-step rules have been applied along a branch.

Note that the idea of deterministically constructing big-step rules can also be exploited in forward proof search, in that the big-step rules described above can take the place of left rules in the inverse method. The main advantages here are that the big-step rules are more relevant to proof search for a given query formula, and that the number of intermediate sequents added to the knowledge base during proof search is reduced, since no focused sequents need to be maintained.

6 Bidirectional Natural Deduction Method

In the backward bidirectional sequent calculus method, we construct big-step rules to conceal all required focused threads. Notice that the focused threads of $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ correspond naturally to segments of $\mathbf{NJ}^{\mathbf{N}}_{\mathbf{IS4}}$ proofs consisting of elimination rule applications, that is, $\downarrow$ sequents. The beginnings of focused threads, where formulas are placed into the stoup, correspond to *reversing rules* in $\mathbf{NJ}^{\mathbf{N}}_{\mathbf{IS4}}$. These are the $\uparrow\downarrow$ rule, as well as all elimination rules with $\uparrow$ sequents as their conclusions. The ends of focused threads, on the other hand, where the stoup formula is dropped into a context, correspond to using a hypothesis with applications of hyp_1 or hyp_2.

This means that the process of building a big-step $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ rule in a top-down way corresponds to building a natural deduction big-step rule by beginning with an application of a reversing rule, and growing it upwards until we reach a leaf. Just as the construction of big-step rules in the sequent calculus is determined uniquely by the form of the query formula, so these natural deduction big-step rules can be deterministically constructed before proof search even begins.

This approach is best demonstrated by an example such as the one given in Sect. 5. For instance, given the pair $L_4^{\approx}$ and L_8^{+} from that example, we begin with the coercion

$$\frac{\Delta; \Gamma \vdash L_4^{\approx} \downarrow}{\Delta; \Gamma \vdash L_8^{+} \uparrow}\uparrow\downarrow$$

Since the immediate parent of $L_4^{\approx}$ in the signed subformula hierarchy is $L_2^{\approx}$, denoting $A \supset B$, the rule application above this coercion must be an application of $\supset$E:

$$\dfrac{\Delta; \Gamma \vdash L_2^{\approx} \downarrow \quad \Delta; \Gamma \vdash L_3^{+} \uparrow}{\dfrac{\Delta; \Gamma \vdash L_4^{\approx} \downarrow}{\Delta; \Gamma \vdash L_8^{+} \uparrow} \uparrow\downarrow} \supset\mathrm{E}$$

The focused thread continues along the the first premise, but the parent of $L_2^{\approx}$ is $L_2^{=}$, indicating the end of this focused thread by an application of init_2:

$$\dfrac{\dfrac{L_2^{=} \in \Delta}{\Delta; \Gamma \vdash L_2^{\approx} \downarrow}\mathrm{init}_2 \quad \Delta; \Gamma \vdash L_3^{+} \uparrow}{\dfrac{\Delta; \Gamma \vdash L_4^{\approx} \downarrow}{\Delta; \Gamma \vdash L_8^{+} \uparrow} \uparrow\downarrow} \supset\mathrm{E} \qquad \rightsquigarrow \qquad \dfrac{\Delta, L_2^{=}; \Gamma \vdash L_3^{+} \uparrow}{\Delta, L_2^{=}; \Gamma \vdash L_8^{+} \uparrow}(2)$$

In similar constructions, $L_1^{\sim}$, denoting $\Box(A \supset B)$, and the pair $L_7^{\sim}$, L_3^{+} produce the natural deduction big-step rules

$$\dfrac{\dfrac{L_1^{-} \in \Gamma}{\Delta; \Gamma \vdash L_1^{\sim} \downarrow}\mathrm{init}_1 \quad \Delta, L_2^{=}; \Gamma \vdash M^{+} \uparrow}{\Delta; \Gamma \vdash M^{+} \uparrow} \Box\mathrm{E} \qquad \rightsquigarrow \qquad \dfrac{\Delta, L_2^{=}; \Gamma, L_1^{-} \vdash M^{+} \uparrow}{\Delta; \Gamma, L_1^{-} \vdash M^{+} \uparrow}(3)$$

and

$$\dfrac{\dfrac{L_7^{-} \in \Gamma}{\Delta; \Gamma \vdash L_7^{\sim} \downarrow}\mathrm{init}_1}{\Delta; \Gamma \vdash L_3^{+} \uparrow} \uparrow\downarrow \qquad \rightsquigarrow \qquad \dfrac{}{\Delta; \Gamma, L_7^{-} \vdash L_3^{+}}(1)$$

The rest of the proof then uses only these big-step rules and introduction rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{L_2^{=}; L_1^{-}, L_7^{-} \vdash L_3^{+} \uparrow}(1)}{L_2^{=}; L_1^{-}, L_7^{-} \vdash L_8^{+} \uparrow}(2)}{L_2^{=}; L_1^{-} \vdash L_6^{+} \uparrow}\supset\mathrm{I}}{L_2^{=}; L_1^{-} \vdash L_5^{+} \uparrow}\Diamond\mathrm{I}}{\cdot; L_1^{-} \vdash L_5^{+} \uparrow}(3)}{\cdot; \cdot \vdash L_0^{+} \uparrow}\supset\mathrm{I}$$

In general, the approach for constructing natural deduction big-step rules is analogous to the method for the backward bidirectional sequent calculus, only turned upside-down, in the sense that the rule at the beginning of an $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ focused thread determines the reversing rule at the bottom of the natural deduction focused thread, while the final application of init dictates the "principal formula" of the ensuing big-step natural deduction rule.

Formula	Size	Modalities	Provable	Histories Time	Inverse Time	Inverse Rules	Bidirectional Time	Bidirectional Rules
32	49	0	N	> 1000	1.36	33	0.01	33
36	175	0	Y	0.08	> 1000	159	> 1000	592
37	68	9	Y	84.79	1.18	60	< 0.01	28
39	42	3	N	8.46	1.83	31	< 0.01	15
44	49	14	Y	75.13	> 1000	51	37.11	21
50	44	7	Y	7.38	> 1000	49	48.76	25

Table 1. Selection of experimental results.

Proof search over natural deductions can then be performed in a backward direction. The only nondeterminism is in whether to apply a big-step rule or an introduction rule, the premises of which are uniquely determined by their conclusions. Note that to guarantee termination, we again disallow using a big-step rule more than once along any branch of a proof.

7 Experimental Results

While benchmark formulas are available for intuitionistic propositional logic and classical modal logics, we are not aware of any benchmark libraries specific to propositional **IS4**. In order to evaluate the performance of our bidirectional approach, we put together a benchmark set of 50 formulas for **IS4**, mostly problems from Raths et al.'s Intuitionistic Logic Theorem Proving (ILTP) library [17] to which we introduced modalities, making them specific to **IS4**. Our benchmark set is available at `http://www.cs.mcgill.ca/~sheila1/is4/`

We have implemented three **IS4** decision procedures in SML: (1) a sequent calculus-based backward prover with a history mechanism for loop-detection, (2) a conventional inverse method prover without big-step rules, and (3) our bidirectional natural deduction prover. The loop-detection prover uses $\mathbf{MJ_{IS4}}$ as a basis for backward proof search, maintaining two histories to detect repeated sequents. One, the modal history, identifies repeated applications of the modal rules $\Box$R and $\Diamond$L, while the other identifies applications of the other rules in between these modal rule applications. This approach is a generalization and extension of Howe's decision procedure [11]. The inverse method prover is based directly on $\mathbf{MJ^{F}_{IS4}}$ but operates on subformulas rather than subformula occurrences. Note that the behaviour of our backward bidirectional sequent calculus prover corresponds exactly to that of the bidirectional natural deduction prover, so we have only implemented the more elegant natural deduction prover.

On many of the smaller problems, there was little measurable difference in the performance of the provers, but some of the problems that did ellicit noticeably different performances are highlighted in table 1. The size column shows the complexity of each formula, computed inductively in the usual way, while the modalities column shows the number of modal operators. Times are in seconds.[1] For the inverse method and bidirectional provers, we show the number of inference rules generated (big-step rules in the case of the bidirectional prover).

As our results demonstrate, the bidirectional natural deduction prover is a competitive alternative to the more conventional provers, equalling or outperforming them on most problems. Comparing the average proving time for problems that were solved, it is far superior. We detected, however, two pathological formulas on which our bidirectional prover was significantly outperformed (formulas 36 and 50 in table 1). Interestingly, there is not always a clear connection between the number of big-step rules generated and the time required to solve a problem in the bidirectional prover. Presumably, the problematic cases were those whose big-step rules where the shortest and thus least useful.

8 Related and Future Work

Although **IS4** has undergone thorough proof-theoretical studies, there has been little work in developing proof search strategies specific to it. We have presented a comprehensive proof-theoretical study of proof search formalism for **IS4**, highlighting the duality between backward and forward search. Moreover, we have demonstrated how to combine the benefits of both to yield bidirectional decision procedures based on sequent calculi and natural deduction. Our experimental results reveal that combining the two traditionally disparate paradigms can be fruitful. While our implementations are naive and incorporate no optimizations, we hope that our results might encourage further study of bidirectional proof search, particularly in other logics.

For instance, in the contextual modal logic of Nanevski, Pfenning, and Pientka [14], structural modality is generalized by relativizing the validity judgement and the modal operators. The techniques discussed in this paper extend very naturally to contextual modal logic, yielding sequent calculi suitable for both backward and forward proof search, but the exact nature of how the generalization to contextual modal logic affects proof search is yet to be explored. The reconciliation of forward and backward proof search has recently been investigated by Chaudhuri and Pfenning [3], who, in the context of linear logic, propose a focusing inverse method prover incorporating big-step rules constructed in a backward way and searched over in a forward direction, opposite to our approach. Unlike their work, which builds on Andreoli's focusing property [1], we extend ideas by Dyckhoff and Pinto [6] and develop a focusing calculus which directly gives rise to a bidirectional natural deduction proof search procedure

In the future, we plan to explore extensions to the first-order case. Although the idea of big-step rules extends, in principle, to first-order quantifiers, the constructed big-step rules become parametric in terms. The useful property of $\mathbf{MJ}^{\mathbf{F}}_{\mathbf{IS4}}$ that eliminated the need for loop-detection in our bidirectional method now only holds for particular instantiations of the terms of the parametric big-step rules. Unfortunately, requiring the storage of rule instantiations reintroduces

[1] All timing results were obtained on a Pentium III 850MHz with 256MB of RAM, running SML/NJ 110.60.

a form of loop-detection. How to overcome this problem and what the proof-theoretical relationship between first-order bidirectional decision procedures and natural deduction provers is remains to be investigated.

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