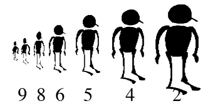
# Priority queue ADT
# Heaps

Lecture 21

## Priority queue ADT

- Like a dictionary, a priority queue stores a set of pairs (key, info)
- The rank of an object depends on its priority (key)

Rear of queue



Front of queue

key: 9 8 6 5 4 2

- Allows only access to
  - Object findMin() //returns info of smallest key
  - Object removeMin() // removes smallest key
  - void insert(key k, info i) // inserts pair
- Applications: customers in line, Data compression, Graph searching, Artificial intelligence…

## Priority queue ADT

| | | | | |
|---|---|---|---|---|
| (**4**,$O_4$) | (**5**, $O_5$) | (**8**,$O_8$) | | |

insert(9,$O_9$)

| | | | | |
|---|---|---|---|---|
| (**4**,$O_4$) | (**5**, $O_5$) | (**8**,$O_8$) | (**9**,$O_9$) | |

remove()

| | | | | |
|---|---|---|---|---|
| (**5**, $O_5$) | (**8**,$O_8$) | (**9**,$O_9$) | | |

insert(6,$O_6$)

| | | | | |
|---|---|---|---|---|
| (**5**, $O_5$) | (**6**,$O_6$) | (**8**,$O_8$) | (**9**,$O_9$) | |

insert(2,$O_2$)

| | | | | |
|---|---|---|---|---|
| (**2**, $O_2$) | (**5**, $O_5$) | (**6**,$O_6$) | (**8**,$O_8$) | (**9**,$O_9$) |

## Implementation of priority queue

Unsorted array of pairs (key, info)

findMin(): Need to scan array O(n)

insert(key, info): Put new object at the end O(1)

removeMin(): First, findMin, then shift array O(n)

Sorted array of pairs (key, info)

findMin(): Just return first element O(1)

insert(key, info):

Use binary-search to find position of insertion. O(log n)

Then shift array to make space. O(n)

## Implementation of priority queue

Using a sorted doubly-linked list of pairs (key, info)

**findMin():** Return first element O(1)

**insert(key, info):**

First, find location of insertion.

Binary Search?

Slow on linked list.

Instead, we have to scan array O(n)

Then insertion is easy O(1)

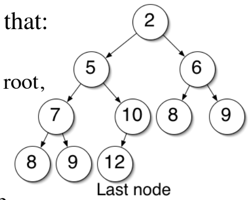
removeMin(): Remove first element of list O(1)

## Heap data structure

- A heap is a data structure that implements a priority queue:
  - findMin(): O(1)
  - removeMin(): O(log n)
  - insert(key, info): O(log n)
- A heap is based on a binary tree, but with a different property than a binary search tree
- heap ≠ binary *search* tree

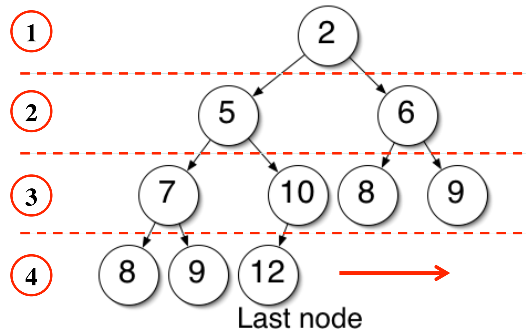
## Heap - Definition

- A **heap** is a binary tree such that:




  - For any node *n* other than the root, key(n) ≥ key( parent(n) )
  - Let h be the height of the heap
    - First h-1 levels are full: For i = 0,…,h-1, there are $2^i$ nodes of depth i
    - At depth h, the leaves are packed on the left side of the tree

## Heap - Example




## Height of a heap

What is the maximum number of nodes that fits in a heap of height h?

$$\sum_{k=0}^{h} 2^k = 2^{h+1} - 1$$

What is the minimum number?

$$(2^h - 1) + 1 = 2^h$$

Thus, the height of a heap with n nodes is:

$$\lfloor \log(n) \rfloor$$

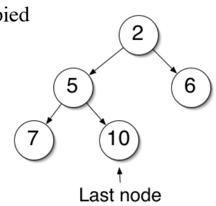
## Heaps: findMin()

The minimum key is always at the root of the heap!

## Heaps: Insert

Insert(key k, info i). Two steps:

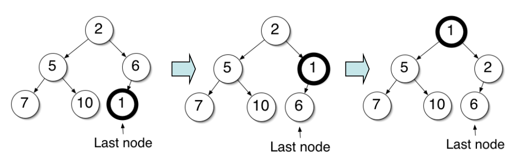
1. Find the left-most unoccupied node and insert (k,i) there temporarily.
2. Restore the heap-order property (see next)




## Heaps: Bubbling-up

Restoring the heap-order property:

– Keep swapping new node with it's parent as long as it's key is smaller than it's parent's key




Running time? $O(h) = O(\log(n))$

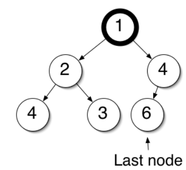
# Insert pseudocode

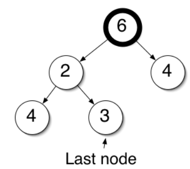
**Algorithm** insert(key k, info i)
**Input**: The key k and info i to be added to the heap
**Output**: (k,i) is added

lastNode ← nextAvailableNode(lastNode)
lastNode.key ← k,     lastNode.info ← i
n ← lastnode
**while** (n.getParent()!=null **and** n.getParent().key > k) **do**
    swap (n.getParent(), n)

# Heaps: RemoveMin()

- The minimum key is always at the root of the heap!
- Replace the root with last node

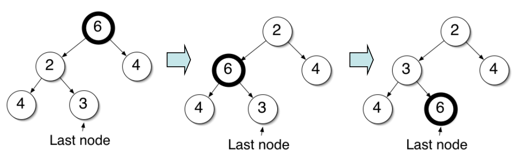



- Restore heap-order property (see next)

# Heaps: Bubbling-down

Restoring the heap-order property:
- Keep swapping the node with its smallest child as long as the node's key is larger than it's child's key




Running time?    $O(h) = O(\log(n))$

# removeMin pseudocode

**Algorithm** removeMin()
**Input**: The key k and info I to be added to the heap
**Output**: (k,i) is added

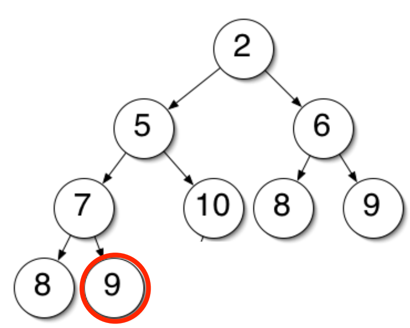
swap(lastNode, root)
Update lastNode
n ← root
**while** (n.key > min(n.getLeftChild().key, n.getRightChild().key)) **do**
    **if** (n.getLeftChild().key < n.getRightChild().key) **then**
        swap(n, n.getLeftChild)
    **else** swap(n, n.getRightChild)

# Finding nextAvailableNode

nextAvailableNode(lastNode) finds the location where the next node should be inserted. It runs in time O(n).

```
n <-- lastNode;
while (n is the right child of its parent && n.parent!=null) do
        n <-- n.parent
if ( n.parent == null ) then nextAvailableNode is the left child of
        the leftmost node of the tree
else
    n <-- n.parent        // go up one more level
    if ( n has no right child) then nextAvailableNode is the right
        child of n
    else
        n <-- n.rightChild   // go down the right child
        while (n has a left child ) do    n <-- n.leftChild
        nextAvailableNode is the left child of n
```

# NextAvailableNode - Example

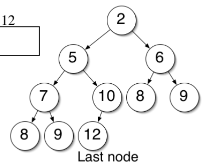


# Array representation of heaps

- A heap with n keys can be stored in an array of length n+1

0 1 2 3 4 5 6 7 8 9 10 11 12

- | 2 | 5 | 6 | 7 | 10 | 8 | 9 | 8 | 9 | 12 |  |




- For a node at index i,
  - The parent (if any) is at index $\lfloor i/2 \rfloor$
  - The left child is at index 2*i
  - The right child is at index 2*i + 1
- lastNode is the first empty cell of the array. To update it, either add or subtract one

# HeapSort

**Algorithm** heapSort(array A[0…n-1])
Heap h ← new Heap()
**for** i=0 **to** n-1 **do**
    h.insert(A[i])
**for** i=0 **to** n-1 **do**
    A[i] ← h.removeMin()

Running time: O(n log n) in worst-case
Easy to do in-place: Just use the array A to store the heap