Induction proofs - Part II

Geometric series: $1+a+a^2+a^3+\dots+a^n$

Claim: $= \dfrac{1-a^{n+1}}{1-a}$

for all $n \geq 0$ integer

for all $a > 0$ real, $a \neq 1$

Example: $a=3$, $n=4$

$1+3^1+3^2+3^3+3^4 = 1+3+9+27+81 = 121$

$$\frac{1-3^{4+1}}{1-3} = \frac{1-3^5}{1-3} = \frac{1-243}{1-3} = \frac{-242}{-2} = 121$$

Proof: $\boxed{1+a+a^2+\dots+a^n = \dfrac{1-a^{n+1}}{1-a}} \rightarrow P(n)$

Base case: $n=0$: LHS $=1$

RHS: $\dfrac{1-a^{0+1}}{1-a} = \dfrac{1-a}{1-a} = 1$

LHS = RHS

Induction step: Assume $P(k)$ is true, i.e.

$$1+a+\dots+a^k = \frac{1-a^{k+1}}{1-a}$$

Goal: Show that $P(k+1)$ is true, i.e.

$$\boxed{1+a+\dots+a^{k+1} = \frac{1-a^{(k+1)+1}}{1-a}}$$

$$\underbrace{1+a+a^2+\dots+a^k}_{\text{by I.H.}}+a^{k+1} = \frac{1-a^{k+1}}{1-a} + a^{k+1}$$

$$= \frac{1-a^{k+1}+(1-a)a\ \dots}{1-a}$$

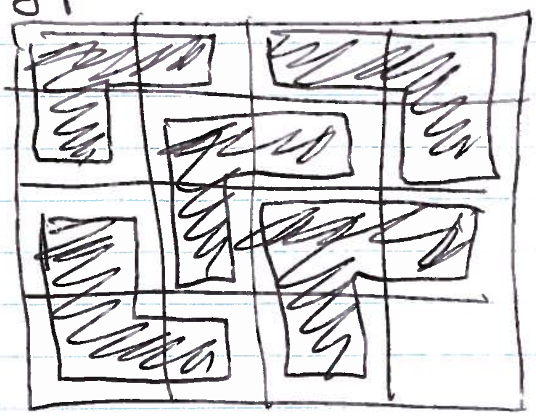
$$= \frac{1-\cancel{a^{k+1}}+\cancel{a^{k+1}}-a\ \dots}{1-a}$$

$$= \frac{1-a^{k+2}}{1-a}$$

$$= \frac{1-a^{(k+1)+1}}{1-a}$$

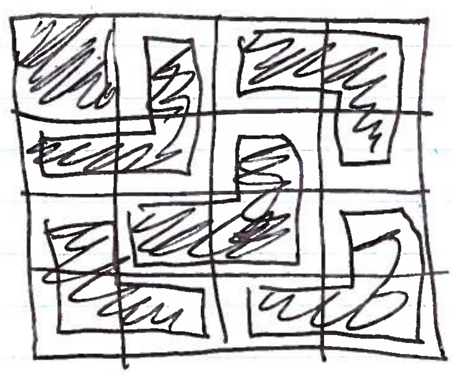
$\Longrightarrow$ $P(n)$ holds for all $n \geq 0$

Tiling problem:



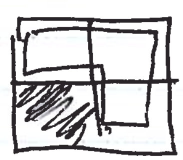
Goal: Cover board with trominoes: 

Consider a $2^n \times 2^n$ board where one square is already covered.
Goal: Cover the rest with trominoes



Proof by induction:

$P(n)$: $\boxed{\text{Any } 2^n \times 2^n \text{ board with one square covered already can be tiled with trominoes, for } n \geq 1}$

Base case: $n=1$ 

No matter where covered sq. is, the rest is a tromino

$$\underbrace{1+a+a^2+\dots+a^k}_{\text{by I.H.}}+a^{k+1} = \frac{1-a^{k+1}}{1-a} + a^{k+1}$$

$$= \frac{1-a^{k+1} + (1-a)a\ldots}{1-a}$$

$$= \frac{1-\cancel{a^{k+1}}+\cancel{a^{k+1}} - a\ldots}{1-a}$$

$$= \frac{1-a^{k+2}}{1-a}$$

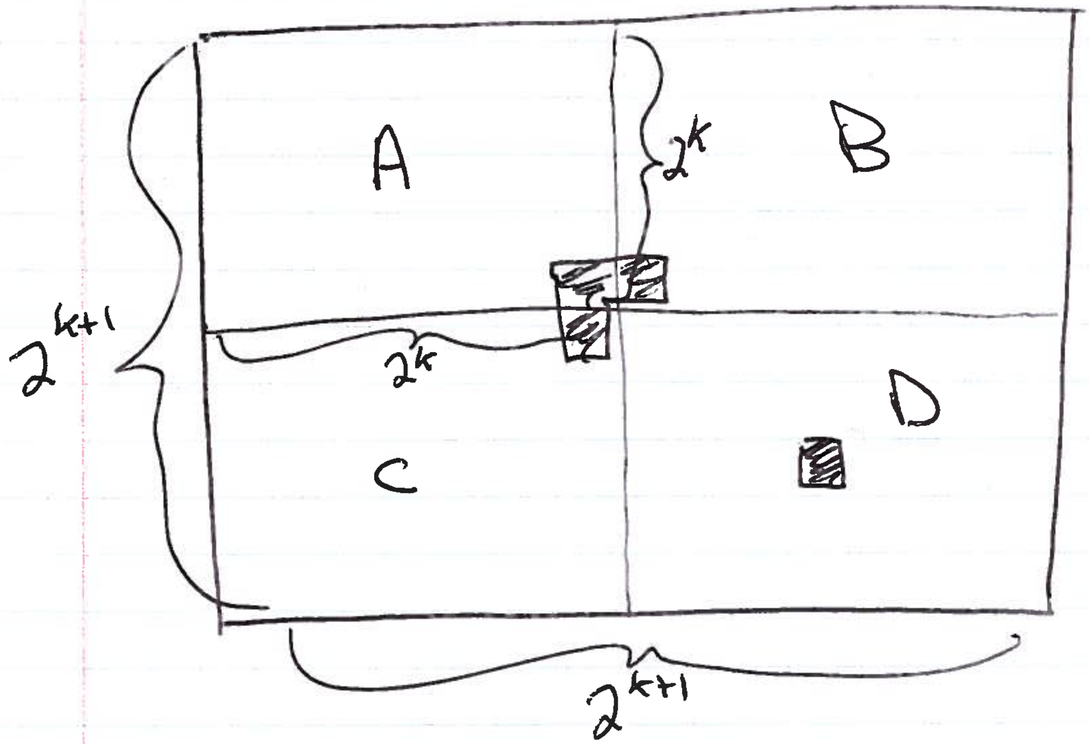
$$= \frac{1-a^{(k+1)+1}}{1-a}$$

$\Longrightarrow$ $P(n)$ holds for all $n \geq 0$

② Induction: Assume $\underbrace{P(k)}$ is true

→ $2^k \times 2^k$ board w… one sq. corner c… tiled

Goal: Show $P(k+1)$ is true



① Sub-board D has one square covered and is of size $2^k \times 2^k$

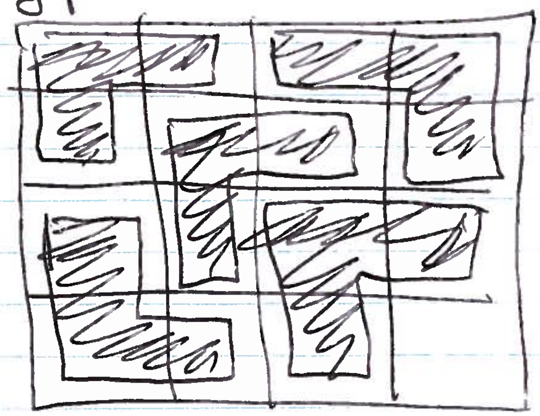
⟹ Because of I.H. D can be t…

② Place tromino in center, to cover one square of A, B, C

Now, A, B, C ~~all~~ have one sq. covered and they are each of size $2^k \times 2^k$

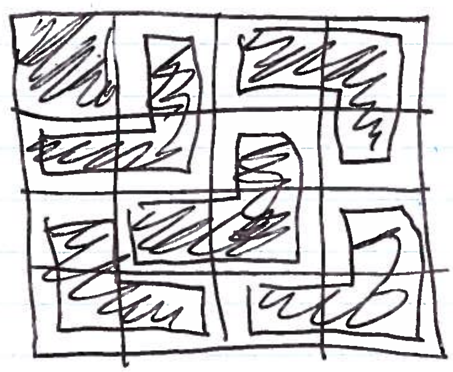
⟹ Because of I.H. the rest of A, B, and C can be tiled with trominoes

Tiling problem:



Goal: Cover board with trominoes: 

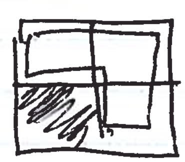
Consider a $2^n \times 2^n$ board where one square is already covered.
Goal: Cover the rest with trominoes



Proof by induction:

$P(n)$ Any $2^n \times 2^n$ board with one square covered already can be tiled with trominoes, for $n \geq 1$

Base case: $n=1$



No matter where covered sq. is, the rest is a tromino

$$\underbrace{1+a+a^2+\dots+a^k}_{\text{by I.H.}}+a^{k+1} = \frac{1-a^{k+1}}{1-a} + a^{k+1}$$

$$= \frac{1-a^{k+1} + (1-a)a}{1-a}$$

$$= \frac{1-\cancel{a^{k+1}} + \cancel{a^{k+1}} - a}{1-a}$$

$$= \frac{1-a^{k+2}}{1-a}$$

$$= \frac{1-a^{(k+1)+1}}{1-a}$$

$\Longrightarrow$ $P(n)$ holds for all $n \geq 0$

Prove:

$P(n)$ $\boxed{n+1 < n}$ for all n

Induction proof:

Assume $P(k)$ is true, i.e. $k+1 < k$

Goal: Show that $P(k+1)$ is true, i.e. $(k+1)+1 < k+1$

$\underbrace{(k+1)}_{<k}+1 < k+1$ (because of I.H)

$\Longrightarrow$ If $P(k)$ is true, then $P(k+1)$ is true

Problem: No base case

~~Proof~~ Claim: $8^n - 3^n$ is divisible by 5 for any $n \geq 1$

Ex: if $n=2$, $8^2 - 3^2 = 64 - 9 = 55$

Proof: Base case: For $n=1$, $8^1 - 3^1 = 5$

Induction step: Assume that $8^k - 3^k$ is divisible by 5

$8^k - 3^k = 5 \cdot \underset{\substack{\uparrow \\ \text{integer}}}{a}$

Goal: $8^{k+1} - 3^{k+1}$ is divisible by 5