

# Finding All Useless Arcs in Directed Planar Graphs

Jittat Fakcharoenphol\*      Bundit Laekhanukit†      Pattara Sukprasert‡

March 13, 2017

## Abstract

Maximum flow is a fundamental problem in Combinatorial Optimization that has numerous applications in both theory and practice. In this paper, we study the *flow network simplification* problem, which asks to remove all the *useless arcs* from the graph. To be precise, an arc is *useless* if it does not participate in any simple  $s, t$ -path. Weihe [FOCS'94, JCSS'97] showed that if there exists an  $O(n \log n)$ -time algorithm for simplifying a flow network, then a maximum  $s, t$ -flow in directed planar graphs can be computed in  $O(n \log n)$ -time. However, there was no known algorithm that could determine all the useless arcs in  $O(n \log n)$ -time. Although an  $O(n \log n)$ -time algorithm for computing maximum flow on planar directed graphs without simplifying a flow network has been later discovered by Borradaile and Klein [SODA'06, J.ACM'09], it remains open whether a directed planar flow network can be simplified in  $O(n \log n)$ -time.

Here we present an algorithm that determines all the useless arcs in  $O(n \log n)$ -time, thus completing the framework of Weihe. Our algorithm improves upon the previous best running time of  $\tilde{O}(n^2)$  for removing all the useless arcs by Misiolek and Chen [COCOON'05, IPL'06] and by Biedl, Brejová and Vinar [MFCS'00]. Our main algorithm requires the planar embedding to contain no clockwise cycle, and every vertex except source and sink has degree three. The bottleneck of our algorithm is the  $O(n \log n)$  time algorithm for preprocessing the planar embedding while all the other parts runs in linear time.

## 1 Introduction

The *maximum  $s, t$ -flow* problem is a fundamental problem in Combinatorial Optimization that has numerous applications in both theory and practice. A basic instance of maximum flow where the underlying graph is planar has been considered as an important special case and has been studied since 50's in the early work of Ford and Fulkerson [FF56]. Since then, there have been steady developments of maximum flow algorithms on undirected planar graphs. Itai and Shiloach [IS79] proposed an algorithm for the maximum  $s, t$ -flow problem on undirected planar graphs that runs in  $O(n^2 \log n)$  time, and in subsequent works [Has81, Rei83, HJ85, Fre87, KRHS94, HKRS97, INSW11], the running time have been improved to the current best  $O(n \log \log n)$ -time algorithm by Italiano et al. [INSW11].

Another line of research is the study of the maximum  $s, t$ -flow problem in directed planar graphs. The fastest algorithm with the running time of  $O(n \log n)$  is due to Borradaile and Klein [BK09].

---

\*Kasetsart University, Thailand

†Weizmann Institute of Science, Israel. Partially supported by the ISF grant 621/12 and I-CORE grant No. 4/11.

‡Kasetsart University, Thailand and The Optimum Solutions, Singapore

Historically, in 1994, Weihe [Wei97] presented a novel approach that would solve the maximum  $s, t$ -flow problem on directed planar graphs in  $O(n \log n)$  time. However, Weihe’s algorithm requires a preprocessing step that transforms an input graph into a particular form: (1) each vertex (except source and sink) has degree three, (2) the planar embedding has no clockwise cycle, and (3) every arc must participate in some *simple  $s, t$ -path*.

The first condition can be guaranteed by a simple (and thus linear-time) reduction, and the second condition can be obtained by an algorithm of Khuller, Naor and Klein in [KNK93] which runs in  $O(n \log n)$ -time. Unfortunately, for the third condition, there was no known algorithm that could remove all such “useless” arcs in  $O(n \log n)$ -time. As this problem seems to be simple, this issue had not been noticed until it was pointed out much later by Biedl, Brejová and Vinar [BBV00]. Although an  $O(n \log n)$ -time algorithm for the maximum  $s, t$ -flow problem in directed planar graphs has been devised by Borradaile and Klein [BK09], the question of removing all the useless arcs in  $O(n \log n)$ -time remains unsolved.

In this paper, we study the *flow network simplification* problem, where we are given a directed planar graph  $G = (V, E)$  on  $n$  vertices, a source vertex  $s$  and a sink vertex  $t$ , and the goal is to remove all the arcs  $e$  that are not contained in any simple  $s, t$ -path. One may observe that the problem of determining the usefulness of a single arc involves finding two arc-disjoint paths, which is NP-complete in general directed graphs [FW80]. Thus, detecting all the useless arcs at once is non-trivial. Here we present an  $O(n \log n)$ -time algorithm that determines all the useless arcs, thus solving the problem left open in the work of Weihe [Wei97] and settling the complexity of simplifying a flow network in directed planar graphs.

The main ingredient of our algorithm is a decomposition algorithm that slices the plane graph into small strips with simple structures. This allows us to design an appropriate data structure to handle each strip separately. Our data structure is simple but requires a rigorous analysis of the properties of a planar embedding. We use information provided by the data structure to determine whether each arc  $e$  is contained in a simple  $s, t$ -path  $P$  in  $O(1)$  time, thus yielding a linear-time algorithm. The main difficulty is that we cannot afford to explicitly compute such  $s, t$ -path  $P$  (if one exists) as it would lead to  $O(n^2)$  running time. The existence of any path  $P$  can only be determined implicitly by our structural lemmas.

Our main algorithm runs in linear time. However, it requires the planar embedding (which is assumed to be given together with the input graph) to contain no clockwise cycle. This can be assumed wlog because ones may apply the algorithm of Khuller, Naor and Klein [KNK93] to modify the planar embedding, which costs an extra  $O(n \log n)$  time in the complexity of the algorithm. Moreover, we assume that every vertex that is not source and sink has degree three, which can be done by a standard reduction in linear-time. The above two reductions may affect the usefulness of arcs in the graph, but this is sufficient for the application of the flow-network simplification algorithm in computing maximum  $s, t$ -flow. By standard reductions, we may further assume that the source  $s$  has outdegree one (otherwise, we replace  $s$  by an arc  $ss'$ ), and the sink  $t$  is in the boundary face of the plane graph.

As a final remark, our algorithm runs in **linear-time** given that the graph has been preprocessed by the algorithms in [KNK93], which runs in  $O(n \log n)$ . This is the bottle-neck of our algorithm.

## 1.1 Related Work

Planar duality plays crucial roles in planar flow algorithm developments. Many maximum flow algorithms in planar graphs exploit a shortest path algorithm as a subroutine. However, the  $O(n \log n)$ -

time algorithm for the maximum  $s, t$ -flow problem in directed planar graphs by Borradaile and Klein [BK09], which is essentially a left-most augmenting-path algorithm, is not based on shortest path algorithms. Erickson [Eri10] reformulated the maximum flow problem as a parametric shortest path problem and devised a similar  $O(n \log n)$  algorithm. Borradaile and Harutyunyan [BH13] explored this path-flow duality further and showed a correspondence between maximum flows and shortest paths in directed planar graphs with no restriction on the locations of the source and the sink.

For directed planar graphs with unit capacities, Eisenstat and Klein [EK13] presented a linear time algorithm for finding the max  $s, t$ -flow.

A seminal work of Miller and Naor [MN95] studied flow problems in planar graphs with multiple sources and multiple sinks. The problem is special in planar graphs as one cannot use standard reduction to the single-source single-sink problem without destroying the planarity. Using a wide range of planar graph techniques, Borradaile, Klein, Mozes, Nussbaum, Wulff-Nilsen [BKM<sup>+</sup>11] presented an  $O(n \log^3 n)$  algorithm for this problem.

## 2 Preliminaries

We use the standard terminologies in graph theory. By *degree* of a vertex in a directed graph, we mean the sum of indegree and outdegree. Let  $G$  be an input planar directed graph with  $n$  vertices and  $m = O(n)$  arcs. We assume that a planar embedding of  $G$  is given as input. Let  $s$  and  $t$  be source and sink vertices, respectively. A *strongly connected component* of  $G$  is a maximal subgraph of  $G$  that is strongly connected. For any strongly connected component  $C$  such that  $s, t \notin V(C)$ , a vertex  $v \in V(C)$  is an *entrance* of  $C$  if there exists an  $s, v$ -path  $P \subseteq G$  such that  $P$  contains no vertices of  $C$  except  $v$ , i.e.,  $V(P) \cap V(C) = \{v\}$ . Similarly, a vertex  $v \in V(C)$  is an *exit* of  $C$  if there exists an  $v, t$ -path  $P \subseteq G$  such that  $P$  contains no vertices of  $C$  except  $v$ , i.e.,  $V(P) \cap V(C) = \{v\}$ .

A path  $P \subseteq G$  is a *simple path* if each vertex appears in  $P$  at most once. We say that an arc  $uv$  is a *useful* arc (w.r.t.  $s$  and  $t$ ) if there is a “simple”  $s, t$ -path  $P$  containing  $uv$ . Thus, the  $s, u$  and  $v, t$  subpaths of  $P$  have no common vertices. Otherwise, if there is no simple  $s, t$ -path containing  $uv$ , then we say that  $uv$  is a *useless* arc (w.r.t.  $s$  and  $t$ ). Similarly, a path  $P$  is *useful* (w.r.t.  $s$  and  $t$ ) if there is a simple  $s, t$ -path  $Q$  that contains  $P$ . Note that if a path  $P$  is useful, then all the arcs of  $P$  are useful. However, the converse is not true, i.e., all the arcs of  $P$  are useful does not imply that  $P$  is a useful path.

Throughout this paper, we assume two properties of the input graph  $G$ :

1. There is *no clockwise cycle* in the planar embedding.
2. Every vertex except  $s$  and  $t$  has *degree three*.
3. The source  $s$  is adjacent to only *one outgoing arc*.
4. The sink  $t$  is on the *boundary face*.

### 2.1 Planar Embedding and Basic Subroutines

We assume that the planar embedding is given in the form of an adjacency list sorted in counterclockwise order. To be precise, our input is a table  $\mathcal{T}$  of adjacency list of arcs incident to each vertex  $v \in V(G)$  (i.e.,  $V(G)$  is the index of  $\mathcal{T}$ ) sorted in counterclockwise order. Each adjacency list  $\mathcal{T}(v)$  is a doubly linked-list of arcs having  $v$  as either heads or tails. (i.e., arcs of the forms

$uv$  or  $vw$ ). We can construct from the table  $\mathcal{T}$ , two tables of adjacency list, say  $\mathcal{T}^{in}$  and  $\mathcal{T}^{out}$ , whose entry  $\mathcal{T}^{in}(v)$  (respectively,  $\mathcal{T}^{out}(v)$ ) for  $v \in V(G)$  is a pointer to a doubly linked-list of arcs entering (respectively, leaving)  $v$ . Thus, given an arc  $vw$ , we can query in  $O(1)$  time an arc  $vw'$  next to  $vw$  in the counterclockwise (or clockwise) order. Moreover, by preprocessing, we can query in  $O(1)$  time the *right-most arc* (resp., *left-most arc*) of  $uv$ , which is an arc  $vw$  in the reverse direction that is nearest to  $uv$  in the counterclockwise (resp., clockwise) order. It is easy to see that this preprocessing can be implemented to run in linear time.

Belows are the list of basic subroutines used in our paper.

- **Next( $vw$ ).** Given an arc  $vw \in E(G)$ , return an arc  $vw'$  (an arc adjacent to  $v$ ) next to  $vw$  in the counterclockwise order.
- **Prev( $vw$ ).** Given an arc  $vw \in E(G)$ , return an arc  $vw'$  (an arc adjacent to  $v$ ) before  $vw$  in the counterclockwise order.
- **Right( $uv$ ).** Given an arc  $uv \in E(G)$ , return an arc  $vw$  (an arc adjacent to  $v$ ) which is an arc in the reverse direction nearest to  $uv$  in the counterclockwise order.
- **Left( $uv$ ).** Given an arc  $uv \in E(G)$ , return an arc  $vw$  (an arc adjacent to  $v$ ) which is an arc in the reverse direction nearest to  $uv$  in the clockwise order.

**Left and Right Direction.** Since we are given a planar embedding, we may compare two paths/arcs using the notion of *left to* and *right to*. Given a path  $P$ , we say that an arc  $e = vw$  (resp.,  $e = uv$ ) is *right to*  $P$  if  $v \in P$  and there exist arcs  $av, vb$  in  $P$  such that  $vw$  appears after  $av$  and before  $vb$  in counterclockwise order. The notion *left to* can be defined analogously. Given two paths  $P$  and  $Q$  that have an intersection, we say that  $P$  is *right to*  $Q$  if any arc of  $P$  entering or leaving  $Q$  is *right to*  $Q$ . (Note that if  $P$  and  $Q$  have no intersection, then we cannot say which path is on the left or on the right.)

The subroutines mentioned previously allow us to query in  $O(1)$  time the *right-most* arc  $vw$  of a given arc  $uv$ , which is the arc in the reverse direction nearest to  $uv$  in the counterclockwise order. Consequently, we may define the *right-first-search* algorithm as a variant of the depth-first-search algorithm that chooses to traverse to the right-most (unvisited) arc in each step of the traversal.

Since we assume that the source  $s$  has a single arc  $e$  leaving it, the right-first-search (resp., left-first-search) started from  $s$  is unique and thus well-defined (because every path must start from the arc  $e$ ).

### 3 Overview and Structural Lemmas

In this section, we give an overview of the algorithm. We start by decomposing  $G$  into a set of strongly connected components. Observe that any arc  $uv$  that connects two strongly connected components is useful if and only if  $u$  is reachable from  $s$  and  $t$  is reachable from  $v$ . We are left to work on arcs inside strongly connected components. We work on each strongly connected component independently.

We state the following useful facts.

**Lemma 1.** *Let  $e'$  be any useless arc in  $G$ . Then any useful arc  $e \in E(G)$  is also useful in  $G - \{e'\}$ .*

*Proof.* By the definition of useful arc  $e$ , there exists a simple  $s, t$ -path  $P$  in  $G$  containing  $e$ , and such path  $P$  cannot contain  $e'$ ; otherwise, it would certify that  $e'$  is useful. Therefore,  $P \subseteq G - \{e'\}$ , implying that  $e$  is useful in  $G - \{e'\}$ .  $\square$

**Lemma 2.** *An arc  $e$  is useful if and only if it is contained in some useful path.*

*Proof.* The lemma follows from the definition of useful arcs, i.e., if an arc  $e$  is contained in a useful path  $P$ , then we know that there must exist a simple  $s, t$ -path  $Q$  that contains  $P$  (and thus  $e$ ), meaning that  $e$  is useful. If  $e$  is useful, then we know that it must be contained in a simple  $s, t$ -path  $Q$ , and every subpath of  $P$  is useful by definition.  $\square$

**Lemma 3.** *Let  $C$  be a strongly connected component, and let  $B$  be a set of boundary arcs of  $C$ , then  $B$  forms a counter clockwise cycle.*

*Proof.* Since we assume that the embedding has no clockwise cycle, it suffices to show that  $B$  is a cycle. We prove by a contradiction. Assume that  $B$  is not a cycle. Then we would have two consecutive arcs in  $B$  that go in opposite directions. (Note that the underlying undirected graph of  $B$  is always a cycle.) That is, there must exist a vertex  $u$  with two leaving arcs, say  $uv$  and  $uw$ . Since  $u, v, w$  are in the same strongly connected component, the component  $C$  must have a  $v, u$ -path  $P$  and a  $w, u$ -path  $P'$ . We may assume minimality of  $P$  and  $P'$  and thus assume that they are simple paths. Now  $P \cup \{uv\}$  and  $P' \cup \{uw\}$  forms a cycle, and only one of them can be counterclockwise (since  $uv$  and  $uw$  are in opposite direction), a contradiction.  $\square$

**Lemma 4.** *Let  $u$  and  $v$  be an entrance and an exit of  $C$ , and let  $P_u$  and  $P_v$  be an  $s, u$ -path  $P_u$  and a  $v, t$ -path  $P_v$  that contains no vertices of  $C$  except  $u$  and  $v$ , respectively. Then  $P_u$  and  $P_v$  are vertex disjoint.*

*Proof.* Suppose  $P_u$  and  $P_v$  are not vertex-disjoint. Then the intersection of  $P_u$  and  $P_v$  induces a strongly connected component strictly containing  $C$ . This is a contradiction since  $C$  is a maximal strongly connected subgraph of  $G$ .  $\square$

Observe that  $C$  forms a region on a surface. Our algorithm deals with two cases. The first case is the **outside case** where the source  $s$  lies outside the boundary of  $C$  in the planar embedding. Since  $t$  is assumed to be in the boundary face of the planar embedding, both  $s$  and  $t$  lie outside the boundary of  $C$ . The second is the **inside case** where the  $s$  lies inside the boundary of  $C$ . See Figure 1 for the example of outside and inside cases. It is possible to have many inside cases (see Figure 2).

## 4 Strip Decomposition

The crucial part of our algorithm is the decomposition algorithm that decompose the component  $C$  into *strips*, which is a region of  $C$  in the planar embedding enclosed by two arc-disjoint paths  $U$  and  $F$  that share start and end vertices.

We will present a *strip-decomposition* algorithm that decomposes a given strip  $C_{U,F}$  into a collection  $\mathcal{S}$  of *minimal* strips in the sense that any strip in  $\mathcal{S}$  has no proper subgraph that is a strip. Moreover, any two strips obtained by the decomposition are either disjoint or share only their boundary vertices. We claim that the above decomposition can be done in linear time. After

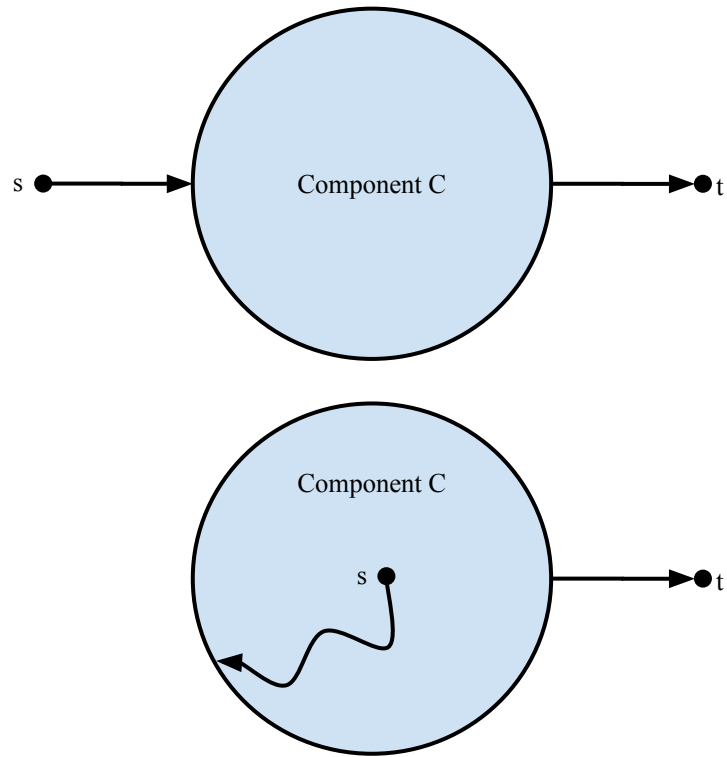


Figure 1: The top figure illustrates the outside case, while the bottom figure illustrates the inside case.

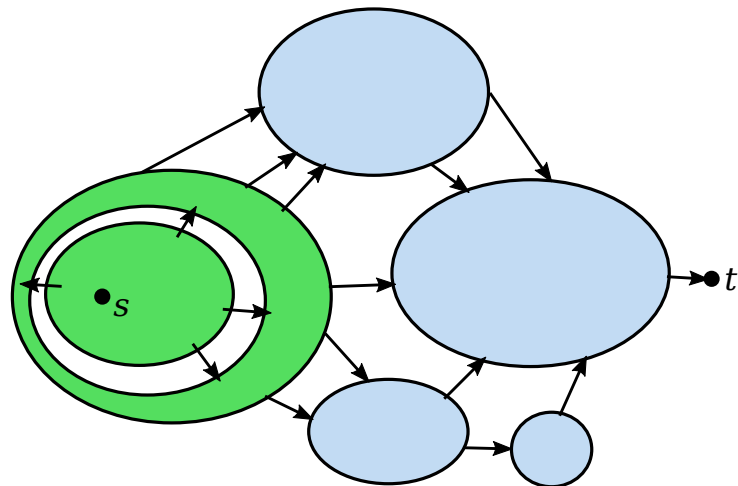


Figure 2: An example showing 6 components. Note the nested inside cases.

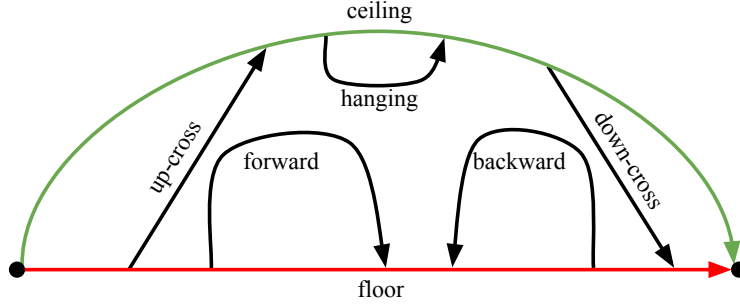


Figure 3: The classification of links in a strip.

the decomposition, all the inner arcs of each strip are useless whereas the boundary arcs are useful provided that the initial floor  $F$  is useful.

Now we formally define *strips*. A *strip* of  $C$ , denoted by  $C_{U,F}$ , is a subgraph of  $C$  enclosed (in the planar embedding) by two arc-disjoint paths  $U$  and  $F$  that share the first and the last vertices. We assume that  $F$  is to the right of  $U$  in the right-first-search order from  $s$ . If we place the first vertex of  $F$  and  $U$  on the left and the last vertex on the right, then  $F$  lies beneath  $U$  in the planar drawing. Thus, we call  $F$  the *floor* and  $U$  the *ceiling* of the strip  $C_{U,F}$ . Note that  $U$  and  $F$  form the boundary of the strip  $C_{U,F}$ .

Consider any strip  $C_{U,F}$ . We call arcs in  $U$  and  $F$  *boundary arcs* and call other arcs (i.e., arcs in  $E(C_{U,F}) - (E(U) \cup E(F))$ ) *inner arcs*. Similarly, vertices in  $U$  and  $F$  are called boundary vertices and other vertices (i.e., vertices in  $V(C_{U,F}) - (V(U) \cup V(F))$ ) are called inner vertices. We call a path  $P \subseteq C_{U,F}$  a *link* in the strip  $C_{U,F}$  if the end-vertices of  $P$  are in the boundary and no other vertices are in the boundary, i.e.,  $V(P) \cap (V(U) \cup V(F)) = \{u, v\}$  where  $u, v$  is the first and last vertices of  $P$ , respectively. Observe that a link has no arcs of the boundary, i.e.,  $E(P) \cap (E(U) \cup E(F)) = \emptyset$ . Let us order vertices on the floor according to the order that they appear in the floor  $F$ , and denote the order of a vertex  $v \in V(F)$  by  $\phi(v)$ . If both end-vertices of the link  $P$  are in  $F$  and  $\phi(u) > \phi(v)$ , then we say that  $P$  is a *backward path* w.r.t to the floor  $F$ . Otherwise, if  $\phi(u) < \phi(v)$ , then we say that  $P$  is a *forward path* w.r.t to the floor  $F$  and ceiling  $C$ . A link that goes from the floor  $F$  to an inner vertex of the ceiling  $U$ , then we call it an *up-cross* path, and if it goes from an inner vertex of the ceiling  $U$  to the floor  $F$ , then we call it a *down-cross* path. A link whose start and end at inner vertices of the ceiling  $C$  is called *hanging path*. Sometime we abuse this classification of paths in  $C_{U,F}$  also for paths that start and end on the floor  $F$ . The classification of these links are shown in Figure 4.

An *extended forward path*  $Q$  is a  $u, v$ -path such that  $u, v \in V(F)$  and  $Q$  contains no backward path. It is easy to see that  $Q$  can be partitioned into a collection of paths  $Q_1, \dots, Q_q$  such that each  $Q_i$  is either a forward path or a subpath of  $F$ . By this definition, the floor  $F$  itself is also an extended forward path. We will sometimes abuse the term extended forward path to mean a forward path. In this sense, the ceiling is also an extended forward path. Observe that any forward path is useful and so are extended forward paths if the floor  $F$  is useful.

**Lemma 5.** *Consider a strip  $C_{U,F}$  such that the source vertex  $s$  and the sink vertex  $t$  are not in  $C_{U,F}$ . If  $F$  is useful, then so is any forward path w.r.t.  $C_{U,F}$ .*

*Proof.* Let  $u$  and  $v$  be the start and end vertices of  $F$  and  $P$  be a forward path connecting  $q, w \in V(F)$ . Since  $F$  is useful, there exists a simple  $s, t$ -path  $R$  that contains  $F$ . We may choose  $R$  to be the shortest such path. Since  $s, t$  are not in  $C_{U,F}$ , vertices in  $V(R) - V(F)$  are not in  $C_{U,F}$ . Let  $R_{s,u}$  and  $R_{v,t}$  be the  $s, u$  and  $v, t$  subpaths of  $R$ . Let  $F_{u,q}$  and  $F_{w,v}$  be the  $u, q$  and  $w, v$  subpaths of  $F$ . Then we construct a new path by concatenating these paths. Let  $R' = R_{s,u} \cdot F_{u,q} \cdot P \cdot F_{w,v} \cdot R_{v,t}$  be such a path. It can be seen that  $R'$  is a simple path. This proves that  $P$  is useful.  $\square$

The following corollary follows immediately from Lemma 5.

**Corollary 6.** *Consider a strip  $C_{U,F}$  such that the source vertex  $s$  and the sink vertex  $t$  are not in  $C_{U,F}$ . If  $F$  is useful, then so is any extended forward path w.r.t.  $C_{U,F}$ .*

*Proof.* We proceed the proof in the same way as in Lemma 5. First, since  $F$  is useful, there exists a simple  $s, t$ -path  $R$  that contains  $F$ . Now consider any extended forward path  $P$ . Observe that  $P$  can be partitioned into subpaths so that each subpath is either contained in  $R$  or is a forward path w.r.t.  $C_{U,F}$ . For each  $u, v$ -subpath  $Q$  of  $P$  that is a forward path, we replace the  $u, v$ -subpath of  $R$  by  $Q$ . Since  $Q$  is arc-disjoint from  $R$ , the above process results in a new simple  $s, t$ -path containing  $P$ . This implies that  $P$  is useful.  $\square$

## 4.1 The Decomposition Algorithm

Now we describe our strip decomposition algorithm. We start our discussion by presenting an abstraction of our algorithm. We denote the initial (input) strip by  $C_{U^*,F^*}$ . We call  $U^*$  the *top-most ceiling* and call  $F^*$  the *lowest floor*. The top-most ceiling  $U^*$  can be a dummy ceiling that does not exist in the graph.

The decomposition algorithm taking as input a strip  $C_{U,F}$ . Then it finds the “right-most” path  $P$  w.r.t. the floor  $F$  that is either a forward path or an up-cross path. This path  $P$  slices the strips into two parts, the *up-strip* and the *down-strip*. See Figure 4. Intuitively, we wish to slice the input strip so that the final collection consists of strips that have no forward, up-cross or down-cross paths. The right-most path guarantees that the down-strip have no down-cross path. Moreover, provided that the initial strip has no down cross path or hanging path, each final strip has no forward, up-cross, down-cross, nor hanging path.

The naive implementation yields an  $O(n^2)$ -time algorithm. We devote the rest of this section to present a linear-time algorithm for the strip-decomposition.

**Lemma 7.** *There is a linear-time algorithm that, given as input a strip  $C_{U^*,F^*}$ , outputs a collection  $\mathcal{S}$  of strips such that each strip  $C_{U,F} \in \mathcal{S}$  has neither forward, up-cross, down-cross nor hanging path. Moreover, if  $F^*$  is useful, then all the arcs in the boundary of any strip  $C_{U,F} \in \mathcal{S}$  are useful.*

## 4.2 Linear Time Implementation

Now we present a linear-time implementation of our strip decomposition algorithm and thus prove Lemma 7.

The input of our algorithm is an initial strip  $C_{U^*,F^*}$ . We insist that the input strip  $C_{U^*,F^*}$  has no hanging path, but it might contain forward, up-cross or down-cross paths. In this sense, we say that a strip is *minimal* if it has neither hanging, forward, up-cross nor down-cross paths (but, the backward paths are allowed). The output of our algorithm is a collection  $\mathcal{S}$  of strips that are



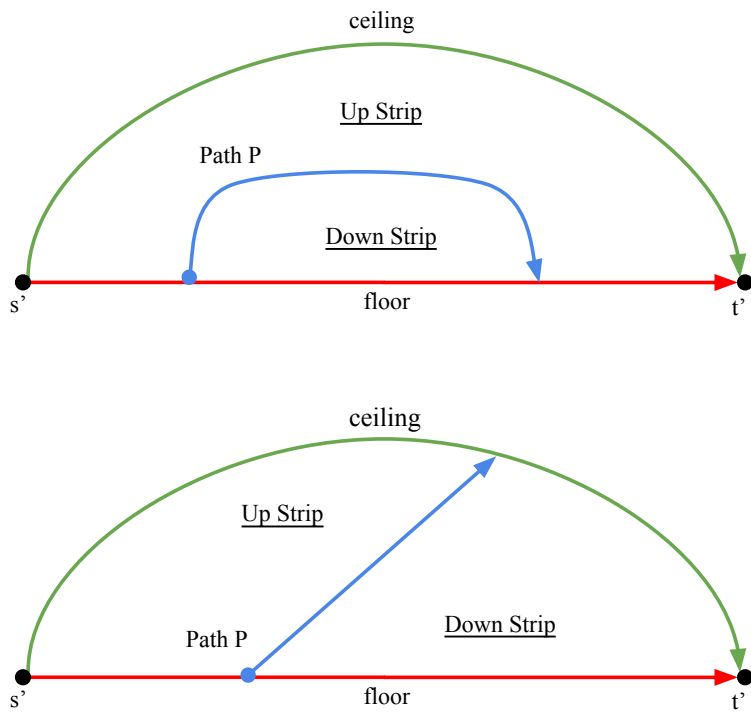


Figure 4: The path  $P$  slices the strip into up-strip and down-strips.

minimal. Let  $s'$  and  $t'$  be the first and the last vertex of  $F$ , respectively. Our Algorithm requires the following initial color settings for vertices.

- All the inner vertices of  $C_{U,F}$  are WHITE.
- All the inner vertices of the ceiling  $U$  are GREEN.
- All the vertices of the floor  $F$  are RED.
- All the inner arcs of  $C_{U,F}$  are WHITE.
- All the border arcs of  $C_{U,F}$  are BLACK.

This initialization is done in Main-Decomposition (Algorithm 1). Then the algorithm calls a subroutine Decompose (Algorithm 3) to decompose the strip  $C_{U,F}$ .

The recursive algorithm Decompose( $C_{U,F}, v$ ) processes the strip in recursive manner. Given a strip  $C_{U,F}$ , the algorithm calls Find-Forward (Algorithm 2) to find a link  $P$  that is either a forward path or an up-cross path, which slices the strip  $C_{U,F}$  into two smaller strips  $C_{U_{down},F_{down}}$  and  $C_{U_{up},F_{up}}$  and recursively partitions these two strips until we have a strip that has neither forward, up-cross nor down-cross paths. Thus, the only links remaining in the strips are backward paths. Our algorithm requires a careful implementation as these strips share boundary arcs. We guarantee that our algorithm visits each arc on the boundary at most  $O(1)$  times and thus the algorithm runs in linear time.

Note that the subroutine decompose will have a glitch if  $v$  is the first vertex  $s'$  of the floor  $F$  and there is no arc between the first arc of the ceiling  $U$  and the first arc of the floor  $F$ . The main algorithm (Algorithm 1) prevents this issue by adding a dummy arc  $(s', z^*)$  between  $(s', a)$  and  $(s', b)$ .

---

**Algorithm 1** Main-Decomposition( $C_{U^*,F^*}$ )

---

- 1: Let  $s', t'$  be the first and the last vertices of  $F^*$ , respectively.
  - 2: Mark all the inner vertices of  $C_{U^*,F^*}$  as WHITE.
  - 3: Mark all the inner vertices of the ceiling  $U^*$  as GREEN.
  - 4: Mark all the vertices of the floor  $F^*$  as RED.
  - 5: Mark  $s'$  as BLACK.
  - 6: Mark all border arcs of  $C_{U^*,F^*}$  as BLACK.
  - 7: Mark all inner arcs of  $C_{U^*,F^*}$  as WHITE.
  - 8:  $\{Glitch\ Prevention\}$
  - 9: **if** there is no arc between the first arc of the ceiling  $U^*$  and the first arc of the floor  $F^*$  **then**
  - 10:   Mark  $s'$  as BLACK.
  - 11:   Let  $v$  be the second vertex of  $F$  (i.e.,  $(s', v) \in E(F)$ ).
  - 12: **else**
  - 13:   Let  $v := s'$ .
  - 14: **end if**
  - 15: Call  $\mathcal{S} := \text{Decompose}(C_{U,F}, v)$ .
  - 16: **return** The collection of minimal strips  $\mathcal{S}$ .
-

---

**Algorithm 2** Find-Forward( $C_{U,F}, uv$ )

---

```
1: Mark  $v$  as GREY.
2: Let  $vw := \text{Right}(uv)$  be the right most arc of  $uv$ .
3: while  $vw$  is WHITE do
4:   Mark  $vw$  as BLACK.
5:   if  $w$  is WHITE then
6:     Call  $P := \text{Find-Forward}(C_{U,F}, vw)$ .
7:     if  $P$  is not an empty path then
8:       return the path  $v \cdot P$ .
9:     end if
10:    if  $w$  is RED or GREEN then
11:      return the path  $(v, w)$ .
12:    end if
13:  end if
14:  Update  $vw := \text{Next}(vw)$ .
15: end while
16: return An empty path  $()$ .
```

---

### 4.3 Implementation Details

Observe that we might create  $O(n)$  strips (and thus have  $O(n)$  floors and ceilings) during the run of the decomposition algorithm. However, we cannot afford to store floors and ceilings of all the strips separately as they are not arc disjoint. Otherwise, just storing them would require a running time of  $O(n^2)$ . To avoid this issue, we keep paths (either floor or ceiling) as doubly linked lists. As we will show in Lemma 12, our decomposition algorithm maintains an invariant that, at any point in time, each arc is contained in at most one floor and at most one ceiling. Consequently, we can store  $C_{U,F}$  by referencing to the first and last arcs of the floor (respectively, ceiling). Moreover, since we store paths as doubly linked list, each *cut-and-join* operations can be done in  $O(1)$  time.

Since we do not have the complete representation of  $C_{U,F}$ , it is not clear if we can check the memberships of arcs in the strip. As such, we use the coloring scheme to mark which vertices are in the ceiling or in the floor. It is more difficult to check if vertices (resp., arcs) are in a particular strip. Thus, we need to describe how to implement Step 9 of Algorithm 1 and Step 5 of Algorithm 3.

Consider a strip  $C_{U,F}$ , and let  $s'$  and  $t'$  be the first and last vertices of the floor  $F$ , respectively. We check if there is an arc between the first arc of the floor  $F$ , say  $s'a$  and the first arc of the ceiling  $U$ , say  $s'b$ , by querying if  $s'b$  is the arc next to  $s'a$  in the counterclockwise order (i.e.,  $\text{Next}(s'a) = s'b$ ). If so, then we know that there can be no arcs between  $s'a$  and  $s'b$ . This gives an implementation of Step 9 of Algorithm 1 that runs in  $O(1)$  time.

Next let  $v$  be a vertex in the floor  $F$ . If  $v = s'$ , then we can check if  $v$  is incident to an inner arc of  $C_{U,F}$  by checking if there is an arc between  $s'a$  and  $s'b$  (as previously discussed). Otherwise, if  $v$  is an inner vertex of the path  $F$ , then we take a subpath  $(u, v, w)$  of  $F$  (which exists because  $v$  is an inner vertex). Then check if the left-most arc of  $uv$  is  $vw$  (i.e.,  $\text{Left}(uv) = vw$ ). If so, then there can be no arc between  $uv$  and  $vw$ , meaning that  $v$  is incident to no inner arcs.

---

**Algorithm 3** Decompose( $C_{U,F}, v$ )

---

- 1: Let  $s'$  and  $t'$  be the first and last vertices of  $F$ , respectively.
- 2: Initial  $P := ()$  as an empty path.
- 3: **while**  $P = ()$  and  $v \neq t'$  **do**
- 4:   Mark  $v$  as BLACK. .
- 5:   **if**  $v \in V(F)$  is incident to an inner arc of  $C_{U,F}$  **then**
- 6:     Let  $w$  be the vertex next to  $v$  in the floor  $F$ .
- 7:     Let  $vz := \text{Next}(vw)$  be the arc next to  $vw$  in the counterclockwise order.
- 8:     Call  $P := \text{Find-Forward}(C_{U,F}, vz)$ . .
- 9:   **end if**
- 10:   Update  $v := w$  (i.e., iterate to the next vertex).
- 11: **end while**
- 12: **If**  $P$  is an empty path, **then** return  $\{C_{U,F}\}$ .
- 13: Mark all the inner vertices of  $P$  as GREEN. .
- 14: Let  $v$  and  $w$  the first and last vertices of  $P$ , respectively.
- 15: **if**  $w$  is RED (i.e.,  $w \in V(F)$ ) **then**
- 16:   Let  $F_{s',v}$ ,  $F_{v,w}$  and  $F_{w,t'}$  be  $s', v, v, w$  and  $w, t'$  subpaths of  $F$ , respectively.
- 17:   Let  $U_{up} := U$  and  $F_{up} := F_{s',v} \cdot P \cdot F_{w,t'}$ .
- 18:   Let  $U_{down} := P$  and  $F_{down} := F_{v,w}$ .
- 19: **else**
- 20:   Let  $F_{s',v}$ ,  $F_{v,t'}$  be  $s', v, v, t'$  subpaths of  $F$ , respectively.
- 21:   Let  $U_{s',w}$ ,  $U_{w,t'}$  be  $s', w, w, t'$  subpaths of  $U$ , respectively.
- 22:   Let  $U_{up} := U_{s',w}$  and  $F_{up} := F_{s',v} \cdot P$ .
- 23:   Let  $U_{down} := P \cdot U_{w,t'}$  and  $F_{down} := F_{v,t'}$ .
- 24: **end if**
- 25: *{ We decompose the down-strip before the up-strip. }*
- 26: Let  $w$  be the vertex next to  $v$  in  $F_{down}$ .
- 27:  $\mathcal{S}_{down} := \text{Decompose}(C_{U_{down}, F_{down}}, w)$ . .
- 28: Mark all the vertices of  $P$  except  $v$  as RED. .
- 29:  $\mathcal{S}_{up} := \text{Decompose}(C_{U_{up}, F_{up}}, v)$ . .
- 30: **return**  $\mathcal{S} := \mathcal{S}_{up} \cup \mathcal{S}_{down}$ .

---

## 4.4 Properties of the Decomposition

Belows are the properties of our algorithms.

**Lemma 8** (Boundary Bounded Search). *Consider a call to  $\text{Find-Forward}(C_{U,F}, uv)$ . Suppose  $v$  is an inner vertex of  $C_{U,F}$ , and each vertex in the boundary of  $C_{U,F}$  is either red, green or black. Then, in any recursive call,  $\text{Find-Forward}$  never visits vertices outside of  $C_{U,F}$ , and never changes colors of border vertices of  $C_{U,F}$ .*

*Proof.* We prove the statement by induction.

Suppose the conditions hold, and consider a call to  $\text{Find-Forward}(C_{U,F}, uv)$ . The only step where the algorithm changes the color of any vertex is in Step 1, where it changes the color of  $v$  to grey. Since  $v$  is an inner vertex of  $C_{U,F}$ , it is clear that the algorithm does not change colors of border vertices in this call. Moreover, all the neighbors of  $v$  must be in  $C_{U,F}$ . This means that the arc  $vw$  that the algorithm processes in the while-loop (Step 3) must be such that  $w$  is in  $C_{U,F}$ , and  $w$  must be white because of the coloring assumption.

Next, consider the recursive-call in Step 6. By the condition in Step 5, the algorithm proceeds to call  $\text{Find-Forward}(C_{U,F}, vw)$  only if  $w$  is white and must be an inner vertex. Thus, the conditions remain hold for the next recursive call, and we deduce inductively that the assertion is true throughout all the remaining recursive calls. Therefore, we conclude that the algorithm never visits vertices outside of the strip  $C_{U,F}$ , and it never changes colors of vertices in the border of  $C_{U,F}$ .  $\square$

**Lemma 9** (Coloring Invariants). *Consider the call to Algorithm 3, say  $\text{Decompose}(C_{U,F}, v)$ . Let  $C_{U,F}$  be the input strip, and let  $s'$  and  $t'$  be the first and the last vertices of the floor  $F$ . Then the following invariants hold at the beginning of the decomposition.*

- All the vertices in  $U$  except  $s'$  and  $t'$  are GREEN.
- All the vertices in the  $s', v$ -subpath of  $F$  except  $v$  are BLACK.
- All the vertices in the  $v, t'$ -subpath of  $F$  except  $v$  are RED.
- All the inner vertices (those in  $V(C_{U,F}) - (V(U) \cup V(F))$ ) are WHITE or GREY.

*Proof.* We prove the invariants by induction. The assertion is trivially true in the initial call to Algorithm 3 (which is in turn called by Algorithm 1).

Suppose the invariants hold until some recursive call to  $\text{Decompose}(C_{U,F}, v)$ . If none of the calls to  $\text{Find-Forward}(C_{U,F}, vw)$  returns a non-empty path  $P$ , then we are done because there would be no more recursive calls. Moreover, no vertices in  $C_{U,F}$  are changed colors except those inner vertices of  $C_{U,F}$  because  $\text{Find-Forward}(C_{U,F}, vw)$  never changes colors of vertices in the floor and the ceiling by Lemma 8.

Now suppose that  $\text{Find-Forward}(C_{U,F}, vw)$  returns a non-empty path  $P$ . By construction, all the vertices in the  $s', v$ -subpath of  $F$  are colored black by Step 4 while other vertices in  $F$  remain red. In Step 13, the Algorithm 3 changes colors of all the inner vertices of  $P$  to green. The last vertex of  $P$  is either red or green, but in both cases, the path  $P$  becomes part of the ceiling  $U_{down}$ , and the coloring invariants remain hold for the pair of ceiling and floor  $U_{down}, F_{down}$  by construction. Thus, we inductively deduce that the invariant holds throughout the recursive call in Step 27. The recursive call may change colors of vertices in the strip  $C_{U_{down}, F_{down}}$ , but it never changes the color of the inner vertices of the ceiling  $U_{down}$ . Thus, in either case, after changing the colors of vertices

in  $P$  except  $v$  to red in Step 28, we must have that the invariants hold for the pair of ceiling and floor  $U_{up}, F_{up}$ . Therefore, inductively, we conclude that the invariants hold throughout the run of the decomposition algorithm.  $\square$

**Lemma 10.** *Prior to the call to  $\text{Find-Forward}(C_{U,F}, vz)$  in Step 8 of Algorithm 3, no out-neighbor of a grey vertex is black or grey. Similarly, if  $\text{Find-Forward}(C_{U,F}, vz)$  returns an empty path, then no out-neighbor of a grey vertex is black or grey.*

*Proof.* We prove the lemma by induction. The assertion is true prior to the first call to Find-Forward because, at the initial call to Algorithm 3 by the main algorithm (Algorithm 1), there is no grey vertex.

Suppose the assertion is true prior to the call to  $\text{Find-Forward}(C_{U,F}, vz)$  in Step 8. We will show that the invariant still hold before to the next call to Find-Forward.

If Find-Forward returns an empty-path, then we know by the stopping condition of the subroutine that it could not find an arc  $vw$  such that  $w$  is a red or green vertex. Thus, all the vertices marked grey by this call to Find-Forward have no red or green out-neighbors. We conclude that the invariant remains hold in the next call to Find-Forward in the while-loop (Step 3).

If Find-Forward returns a non-empty path  $P$ , then let  $\mathcal{Z}$  be a vertex that has been scanned by Find-Forward, i.e., vertices that are colored grey in this call. Observe that vertices in  $\mathcal{Z} - V(P)$  are those that have been scanned prior to vertices in  $V(P)$ . Thus, any vertex in  $\mathcal{Z} - V(P)$  has no white, red nor green out-neighbors; otherwise, Find-Forward would have terminated with a non-empty path. As the algorithm marks all the inner vertices of  $P$  as green (Step 13) the invariant remain holds for the next call to Find-Forward (if one exists) in the recursive call to  $\text{Decompose}(C_{U_{down}, F_{down}}, w)$  (Step 27) regardless of the color of the last vertex of  $P$ . We apply our arguments inductively and conclude that the invariant holds throughout the recursive-call and all its children processes. Then at the time the algorithm returned from  $\text{Decompose}(C_{U_{down}, F_{down}}, w)$  (after backtracking) all the grey vertices have neither white, red nor green out-neighbors. Now, as the algorithm changes only the colors of the vertices in  $V(P) - \{v\}$  (from green to red), the invariant still holds before calling  $\text{Decompose}(C_{U_{up}, F_{up}}, v)$  and thus the next call to Find-Forward (if such call occurs). Therefore, we apply the induction hypothesis and conclude that the invariant holds prior to every call to Find-Forward.  $\square$

**Lemma 11.** *Consider the run of  $\text{Find-Forward}(C_{U,F}, vz)$  in Step 8 of Algorithm 3. Then the algorithm returns a path  $P$  starting from  $v$  that is either a forward path or an up-cross path if one exists. Thus,  $P$  is arc-disjoint from  $U$  and  $F$ .*

*Proof.* We recall to the coloring invariants of Algorithm 3 in Lemma 9. It can be seen by the construction that, at the time Algorithm 3 calls  $\text{Find-Forward}(C_{U,F}, vz)$ , all the vertices in the  $s', v$ -subpaths of  $F$  are black.

Suppose the algorithm finds a path  $P$ , and let  $x$  and  $y$  be the first and last vertex of  $P$ , respectively. Then all the inner vertices of  $P$  must be grey because all the vertices that Find-Forward adds to  $P$  except  $x$  and  $y$  is grey by construction. (Note that  $y$  is red or green by the stopping condition.) Thus,  $P$  must be a link that goes from a black vertex to a red vertex or to a green vertex. We now apply the coloring invariants (Lemma 9). If it is the former case, then  $P$  must go from the  $s', x$ -subpath  $F_{s',x}$  of  $F$  to  $V(F) - V(F_{s',x})$ , meaning that  $P$  is a forward path. If it is the latter case, then  $P$  must go from the floor  $F$  to the ceiling  $U$ , meaning that that  $P$  is an up-cross path. This proves the first statement of the lemma.

Now suppose that Find-Forward terminates without finding any path  $P$  (thus, it returns an empty path). Then we will show that there is no forward nor up-cross path w.r.t.  $C_{U,F}$ . By Lemma 10, we know that, prior to and after the call to Find-Forward( $C_{U,F}, vz$ ), no grey vertices have a path to red or green vertices (because, otherwise, some of them would have white, red or green out-neighbors). Thus, if there is a path  $Q$  in  $C_{U,F}$  that goes from  $v$  to red or green vertices, then all the inner vertices of  $Q$  must be white. Let  $\mathcal{Z}$  be the vertices scanned by Find-Forward( $C_{U,F}, vz$ ) (those that marked grey by the algorithm plus the vertex  $v$ ). Then we know that there is a vertex  $x \in V(Q) \cap \mathcal{Z}$  and a vertex  $y \in V(Q) - \mathcal{Z}$ ; otherwise, the path  $Q$  would have scanned by Find-Forward, and it would have returned a non-empty path. We may assume that  $xy$  is an arc of  $Q$ . Since  $y$  is not scanned by Find-Forward, it must be white while  $x$  is grey. But, then we have a contradiction since a grey vertex  $x$  has a white out-neighbor  $y$ . Hence, if Find-Forward( $C_{U,F}, vz$ ) returns an empty path, there is no path from  $v$  to red or green vertices. This proves the lemma.  $\square$

**Lemma 12** (Properties of the Collection of Strips). *Let  $\mathcal{S}$  be the collection of strips computed by Algorithm 1. Then the following statements hold.*

- Any arc  $e \in E(C)$  is contained in at most one floor and at most one ceiling, i.e., there is at most one strip  $C_{U,F} \in \mathcal{S}$  such that  $e \in E(U)$  (resp.,  $E(F)$ ).
- No two strips share inner arcs, i.e., for any strips  $C_{U,F}, C_{U',F'} \in \mathcal{S}$ ,  $(E(C_{U,F}) - (E(U) \cup E(F))) \cap (E(C_{U',F'}) - (E(U') \cup E(F'))) = \emptyset$ .
- Any strip  $C_{U,F} \in \mathcal{S}$  has no up-cross or forward path.
- Suppose the initial strip  $U^*$  in Main-Decompose (Algorithm 1) has no path outgoing from its inner vertex to  $U$  or  $F$ . Then any strip  $C_{U,F} \in \mathcal{S}$  that has no down-cross path.
- Suppose  $F^*$  is a useful path. Then, for any strip  $C_{U,F} \in \mathcal{S}$ , all the arcs in  $U$  and  $F$  are useful.

*Proof.* We prove the first property by induction. Consider the work of the subroutine Decompose (Algorithm 3). If Decompose( $C_{U,F}, v$ ) terminates without any recursive call, then it returns  $\{C_{U,F}\}$ . Thus, the first property holds. If Decompose( $C_{U,F}, v$ ) makes other recursive calls (i.e., Decompose( $C_{U_{down}, F_{down}}, w$ ) and Decompose( $C_{U_{up}, F_{up}}, v$ )), then we know by the construction of  $U_{down}, U_{up}$  and  $F_{down}, F_{up}$  that each of these pairs are arc disjoint. Thus, the property inductively holds.

To prove the second property, let us consider the work of the decomposition algorithm (Algorithm 3). We will show by reverse induction that a collection of strips returned from each recursive call to Algorithm 3 are disjoint in the inner part (i.e., inner arcs and vertices). This is true when the algorithm cannot find a forward path  $P$ . Inductively, assume that the assertion is true until some recursive call where the algorithm finds a non-empty path  $P$ . By the condition in Step 5, we know that the first arc  $vz$  of  $P$  must be an inner arc, and  $z$  is an inner vertex of  $C_{U,F}$ . Hence, by Lemma 8,  $P$  must be a path in  $C_{U,F}$ . By coloring invariants (Lemma 9), the path  $P$  is either a forward or up-cross path. Thus, in any case,  $P$  slices the strip  $C_{U,F}$  into two regions, which are disjoint in the inner parts. Also, by construction, these two regions are enclosed by the paths  $U_{down}, F_{down}$  and  $U_{up}, F_{up}$ , respectively. Thus, the strips  $C_{U_{down}, F_{down}}$  and  $C_{U_{up}, F_{up}}$  are disjoint in the inner parts, i.e., they have no common inner vertices. Applying the same arguments inductively, we prove the second property.

Now we prove the third property. Suppose for a contradiction that there is a strip  $C_{U,F} \in \mathcal{S}$  that has a forward or up-cross path. Then, by Lemma 11, during the run of the decomposition algorithm (Algorithm 3), Find-Forward would have returned a non-empty path  $P$ , and the algorithm must have decomposed  $C_{U,F}$  into smaller strips  $C_{U_{down},F_{down}}$  and  $C_{U_{up},F_{up}}$ .

Next we prove the fourth property. We suppose that the initial ceiling  $U^*$  in Main-Decompose (Algorithm 1) has no path outgoing from its inner vertex to  $U$  or  $F$ . We will show that all the strips in  $\mathcal{S}$  has no down-cross path. We prove a stronger claim that the condition holds for every call to  $\text{Decompose}(C_{U,F}, v)$  (Algorithm 3). To be precise, consider a call to  $\text{Decompose}(C_{U,F}, v)$ . Assume that no inner vertex of  $U$  has a path to  $U$  or  $F$ . Thus,  $C_{U,F}$  has no down-cross path. If there is no more decomposition, (i.e., Find-Forward never returns a non-empty path), then we are done. Otherwise, if Find-Forward returns a non-empty path  $P$ , then the decomposition algorithm produces two strips  $C_{U_{down},F_{down}}$  and  $C_{U_{up},F_{up}}$ .

Let us first examine the down-strip  $C_{U_{down},F_{down}}$ . If there is a path  $Q$  from the ceiling  $U_{down}$  to the floor  $F_{down}$ , then we know by Lemma 10 that all the inner vertices of  $Q$  must be white. This path  $Q$  must go from a vertex  $x$  in  $V(U_{down}) - V(P)$  to a vertex  $y$  in the floor  $F_{down}$ , which must be a subpath of  $F$  by a construction. Observe that  $Q$  must be right to  $P$  w.r.t. the arc  $vz$  (in the call  $\text{Find-Forward}(C_{U,F}, vz)$ ). But, then Find-Forward must have processed an arc  $xa$  of  $Q$  before processing an arc  $xb$  of  $P$ . Thus,  $a$  cannot be a white vertex because its color must have been changed, a contradiction. (Indeed, Find-Forward would have returned a path different from  $P$ .)

Next we examine the up-strip  $C_{U_{up},F_{up}}$ . We apply similar arguments. Suppose  $C_{U_{up},F_{up}}$  has a path  $Q$  from  $U_{up}$  to  $F_{up}$ . Observe that in any case  $U_{up}$  must be a subpath of the ceiling  $U$ . Since there is no path from an inner vertex of  $U$  to  $U$  or  $F$ , we know that  $Q$  must go from an inner vertex of  $U_{up}$  to a vertex  $x$  of  $P$ . But, then by concatenating  $Q$  with a subpath of  $P$  starting from  $x$ , we must have a path  $Q'$  that goes from  $U$  to either  $U$  or  $F$ , a contradiction. Therefore, we inductively conclude that there is no down-cross path in any strip  $C_{U,F}$  in  $\mathcal{S}$ , proving the fourth property.

Finally, we prove the last property. Consider the construction of the up and down strips during the decomposition of  $C_{U^*,F^*}$ . All the paths  $U_{up}, F_{up}, U_{down}, F_{down}$  are extended forward paths of  $F^*$  and thus useful by Lemma 5. Inductively, we repeat the same arguments on  $C_{U_{up},F_{up}}$  and  $C_{U_{down},F_{down}}$ , we have that the boundary arcs of any strip  $C_{U,F} \in \mathcal{S}$  are useful.  $\square$

**Running Time Analysis.** Now we analyze the running time of our algorithm. We claim that our decomposition algorithm (Algorithm 1) runs in linear time. Our proof requires analyzing how many times we scan each vertex and each arc in the component  $C$ .

**Claim 13.** *The decomposition algorithm scans each arc at most  $O(1)$  times and scans each vertex  $v$  at most  $O(1)$  times the total degree of  $v$  (indegree plus outdegree).*

*Proof.* Consider the run of the algorithm Decompose (Algorithm 3). The algorithm scans an arc in this call only if  $uv$  is an arc the floor  $F$  or is an arc leaving the floor. Since the algorithm keeps moving the pointer to the next arc every time it scans an arc on the floor, the algorithm never scans an arc on the floor twice, and such pointer is passed down to the deeper call, which means that the entire process scans each arc once.

Next consider the run of the algorithm Find-Forward (Algorithm 2). If the algorithm finds a black arc  $vw$ , then it will stop the process and return to the main procedure. Otherwise, it marks  $vw$  as black and scans the arc next to  $vw$  in the counterclockwise order. The algorithm thus scans an arc again only when it has scanned all the arcs leaving  $v$ . Thus, the procedure scans each arc at most twice. In sum, the algorithm scans any arc at most  $O(1)$  times.



Observe that any vertex  $v$  has been marked grey before its leaving arcs are being scanned (and perhaps re-colored as green or red later), any scanning process will stop whenever it reaches  $v$ . After the algorithm scans all the arcs entering  $v$ , the algorithm will never scan  $v$  again, which means that the total time the algorithm scans each vertex  $v$  is  $O(1)$  times the total degree of  $v$  (indegree plus outdegree). This completes the proof.  $\square$

It follows from Claim 13 that the decomposition algorithm scans every element (arcs and vertices) of the component  $C$  at most  $O(1)$  times. This implies a linear running time of the decomposition algorithm. Therefore, Lemma 7 immediately follows from Lemma 12 and the running time analysis.

## 5 The Outside Case: Determining useless arcs when $s$ is not in the component

In this section, we describe an algorithm that determines all useful arcs of a strongly connected component  $C$  when the source  $s$  is in the infinite face.

We first outline our approach. Consider a strongly connected component  $C$ . If the boundary of  $C$  is not a cycle, then the strong connectivity implies that we can extend some path on the boundary to form a clockwise cycle, which contradicts to our initial assumption that the embedding has no clockwise cycle.

If all arcs in the boundary of  $C$  are useless, then no arcs of  $C$  are useful. Otherwise, we claim that some arcs of  $Q$  are useful and some are useless. Moreover, these arcs partition  $Q$  into two paths  $Q_1$  and  $Q_2$  such that  $Q_1$  is a useful path and all arcs of  $Q_2$  are useless.

**Lemma 14.** *Consider the boundary of a strongly connected component  $C$ . Let  $Q$  be the cycle that forms the boundary of  $C$ . Then either all arcs of  $C$  are useless or there are non-trivial paths  $Q_1$  and  $Q_2$  such that*

- $E(Q) = E(Q_1) \cup E(Q_2)$ .
- $Q_1$  is a useful path.
- All arcs of  $Q_2$  are useless.

Moreover, there is a linear-time algorithm that computes  $Q_1$  and  $Q_2$ .

*Proof.* First, we claim that all arcs of  $C$  are useless if and only if  $Q$  has no entrance or has no exit.

If  $Q$  has no entrance or has no exit, then it is clear that all arcs of  $C$  are useless because there can be no simple  $s, t$ -path using an edge of  $C$ .

Suppose  $Q$  has at least one entrance  $u$  and at least one exit  $v$ . We will show that  $Q$  has both useful and useless arcs, which can be partitioned into two paths as in the statement of the lemma.

Observe that no vertices of  $Q$  can be both entrance and exit because of the degree-three assumption. (Any vertex that is both entrance and exit must have degree at least four, two from  $Q$  and two from the entrance and the exit paths.)

For any pair of entrance and exit  $u, v$ , let  $Q_{uv}$  be the  $u, v$ -subpath of  $Q$ . We claim that  $Q_{uv}$  is a useful path. To see this, we apply Lemma 4. Thus, we have an  $s, u$ -path  $P_u$  and a  $v, t$ -path  $P_v$  that are vertex disjoint. Moreover,  $P_u$  and  $P_v$  contain no vertices of  $Q$  except  $u$  and  $v$ , respectively. Thus,  $R = P_u \cdot Q_{uv} \cdot P_v$  form a simple  $s, t$ -path, meaning that  $Q_{uv}$  is a useful path.

Notice that  $Q$  has no useless arc only if  $Q$  has two distinct entrances  $u_1, u_2$  and two distinct exits  $v_1, v_2$  that appear in  $Q$  in interleaving order  $(u_1, v_1, u_2, v_2)$ . Now consider a shortest  $s, u_1$ -path  $P_{u_1}$ , a shortest  $s, u_2$ -path  $P_{u_2}$ , and the  $u_1, u_2$ -subpath  $Q_{u_1, u_2}$  of  $Q$ . These three paths together enclose any path that leaves the component  $C$  through an exit  $v_1$ . Thus, any  $v_1, t$ -path must intersect either  $P_{u_1}$  or  $P_{u_2}$ , contradicting Lemma 4.

Consequently, a sequence of entrances and exits appear in consecutive order on  $Q$ . Let us take the first entrance  $u^*$  and the last exit  $v^*$ . We know from the previous arguments that any pair of entrance and exit yields a useful subpath of  $Q$ . More precisely, the  $u^*, v^*$ -subpath  $Q_1$  of  $Q$  must be useful and must contain all the entrances and exits because of the choices of  $u^*$  and  $v^*$ . Consider the other subpath – the  $v^*, u^*$ -subpath  $Q_2$  of  $Q$ . The only entrance and exit on  $Q_2$  are  $u^*$  and  $v^*$ , respectively. Thus, any path  $s, t$ -path  $P$  that contain an arc  $e$  of  $Q_2$  must intersect with the path  $Q_1$ . But, then the path  $P$  together with the boundary  $Q$  enclose the region of  $C$  that has no exit. Thus,  $P$  cannot be a simple path. It follows that no arcs of  $Q_2$  are useful.

To distinguish the above two cases, it suffices to compute all the entrances and exits of  $Q$  using a standard graph searching algorithm. The running time of the algorithm is linear, and we need to apply it once for all the strongly connected components. This completes the proof.  $\square$

## 5.1 Algorithm for The Outside Case

Now we present our algorithm for the outside case. We assume that some arcs in the boundary of  $C$  are useful and some are useless.

Our algorithm decides whether each arc in  $C$  is useful or useless by decomposing  $C$  into a collection of minimal strips using the algorithm in Section 4. Then any arc enclosed inside a minimal strip is useless (only boundary arcs are useful). Thus, we can determine the usefulness of arcs in each strip. Observe, however, that the component  $C$  is not a strip because it is enclosed by a cycle instead of two paths that go in the same direction. We transform  $C$  into a strip as follows. First, we apply the algorithm as in Claim 14. Then we know the boundary of  $C$ , which is a cycle consists of two paths  $Q_1$  and  $Q_2$ , where  $Q_1$  is a useful path and  $Q_2$  is a useless path. Let  $F^* = Q_1$ , and call it the *lowest-floor*. Note that  $F^*$  starts at an entrance  $s_C$  and ends at an exit  $t_C$ . We add to  $C$  a dummy path  $U = (s_C, u^*, t_C)$ , where  $u^*$  is dummy vertex, and call  $U^*$  the *top-most ceiling*, which is a dummy ceiling. This transforms  $C$  into a strip, denoted by  $C_{U^*, F^*}$ . (See Figure 5)

Now we are able to decompose  $C_{U^*, F^*}$  into a collection  $\mathcal{S}$  of minimal strips by calling `Main-Decompose( $C_{U^*, F^*}$ )`. Since  $F^*$  is a useful path, we deduce from Corollary 6 that all the strips  $C_{U, F} \in \mathcal{S}$  are such that both  $U$  and  $F$  are useful paths.

Next we show that no inner arcs of a strip  $C_{U, F} \in \mathcal{S}$  are useful, which then implies only arcs in the boundaries of strips computed by the strip decomposition are useful. Therefore, we can determine all the useful arc in  $C$  in linear-time.

**Lemma 15.** *Consider a strip  $C_{U, F} \in \mathcal{S}$  computed by the strip-decomposition algorithm. Then no inner arcs  $e$  of  $C_{U, F}$  are useful. More precisely, all arcs in  $e \in E(C_{U, F}) - E(U \cup F)$  are useless.*

*Proof.* Suppose for a contradiction that there is an inner arc  $e$  of  $C_{U, F}$  that is useful. Then there is an  $s, t$  path  $P$  containing  $e$ . We may also assume that  $P$  contains a link  $Q$ . That is,  $Q$  starts and ends at some vertices  $u$  and  $v$  in  $V(U \cup F)$ , respectively, and  $Q$  contains no arcs of  $E(U \cup F)$ . By Lemma 12,  $Q$  must be a backward path. Hence, it suffices to show that any backward path inside the strip  $C_{U, F}$  are useless.

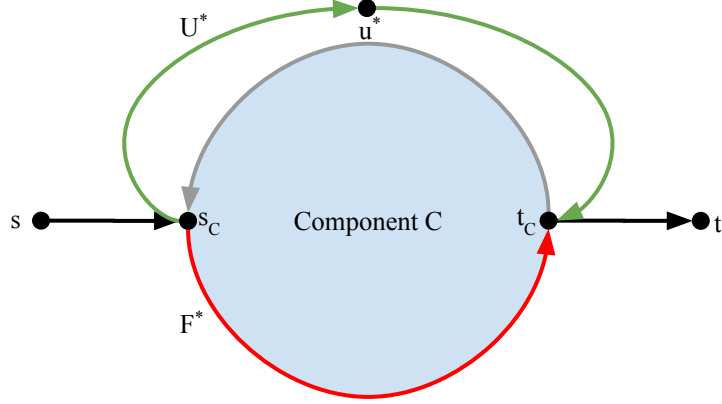


Figure 5: The initial strip before the decomposition.

By the definition of the backward path  $Q$ ,  $v$  must appear before  $u$  on  $F$ , and by Lemma 12,  $F$  is a useful path, meaning that there is an  $s, t$ -path  $R$  containing  $F$ . Let  $s'$  and  $t'$  be the start and end vertices of the strip  $C_{U,F}$  (i.e.,  $s'$  and  $t'$  are the common start and end vertices of the paths  $U$  and  $F$ , respectively). Then we have four cases.

- **Case 1:**  $v = s'$  and  $u = t'$ . In this case, the path  $U$  and  $Q$  form a clockwise cycle, a contradiction.
- **Case 2:**  $v = s'$  and  $u \neq t'$ . In this case,  $v$  has degree three in  $Q \cup U \cup F$ . Thus, the path  $R \supseteq F$  must start at the same vertex as  $R$ ; otherwise,  $v$  would have degree at least four. This means that  $s = s'$ , but then  $P$  could not be a simple  $s, t$ -path because  $s$  must appear in  $P$  at least twice, a contradiction.
- **Case 3:**  $v \neq s'$  and  $u = t'$ . This case is similar to the former one. The vertex  $u$  has degree three in  $Q \cup U \cup F$ . Thus, the path  $R \supseteq F$  must end at the same vertex as  $R$ ; otherwise,  $u$  would have degree at least four. This means that  $t = t'$ , but then  $P$  could not be a simple  $s, t$ -path because  $t$  must appear in  $P$  at least twice, a contradiction.
- **Case 4:**  $u \neq t'$  and  $v \neq s'$ . Observe that  $P$  must enter and leave  $C_{U,F}$  on the right of  $F$  (and thus  $R$ ) at some vertices  $a$  and  $b$ , respectively. We may assume wlog that the  $a, u$ -subpath and the  $v, b$ -subpath of  $P$  are contained in  $R$ . Moreover, it can be seen that  $u$  and  $v$  appear in different orders on the path  $R$  (that contains  $F$ ) and the path  $P$  (that contains  $Q$ ). Thus,  $P$  must intersect either the  $s, b$ -subpath of  $R$ , say  $R_{s,b}$ , or the  $a, t$ -subpath of  $R$ , say  $R_{a,t}$ . (Note that  $R$  contains no inner vertices of  $Q$  by the definition of a backward path.)

Suppose  $P$  intersects  $R_{a,t}$  at some vertex  $x$ , and assume that  $x$  is the last vertex of  $P$  in the intersection of  $P$  and  $R_{a,t}$ . Since  $P$  enters  $R$  at the vertex  $a$  from the right, the  $x, a$ -subpath of  $P$  must lie on the right of  $R$ . But, then the union of the  $x, a$ -subpath of  $P$  and the  $a, x$ -subpath of  $R$  forms a clockwise cycle, a contradiction. (See Figure 6 for illustration.)

Now suppose  $P$  intersects  $R_{s,b}$  at some vertex  $y$ , and assume that  $y$  is the first vertex of  $P$  in the intersection of  $P$  and  $R_{s,b}$ . Since  $P$  leaves  $R$  at the vertex  $b$  from the right, the  $b, y$ -subpath of  $P$  must lie on the right of  $R$ . But, then the union of the  $b, y$ -subpath of  $P$  and the  $y, b$ -subpath of  $R$  forms a clockwise cycle, again a contradiction.

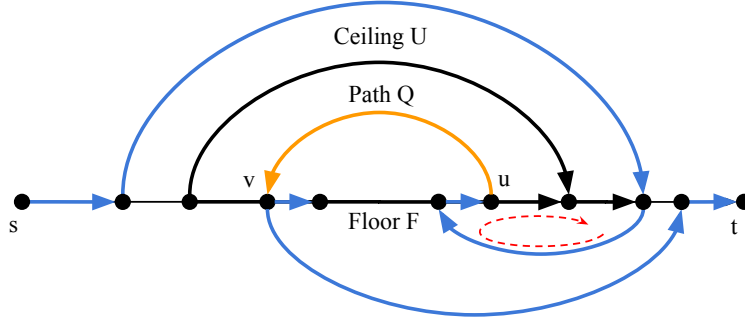


Figure 6: An illustration of Case 4 in the proof of Lemma 15

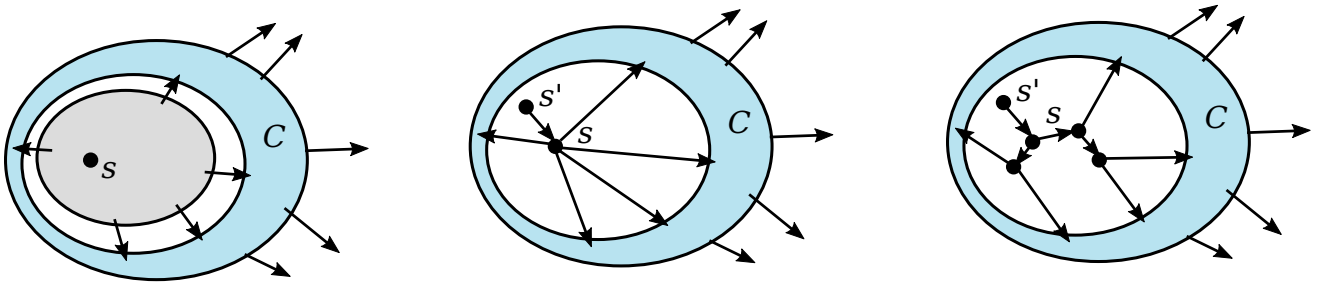


Figure 7: A reduction for simplifying the source structure for the inside case.

Therefore, all backward paths inside the strip  $C_{U,F}$  are useless and so do inner arcs of  $C_{U,F}$ .  $\square$

## 6 The Inside Case: Determining useful arcs of a strongly connected component when $s$ is inside of the component

In this section, we describe an algorithm for the case that the source vertex  $s$  is in the strongly connected component  $C$ . In this case, the source vertex  $s$  is enclosed in some face  $f_s$  in the component  $C$ . It is possible that  $s$  is also enclosed in other strongly connected components (see Figure 2). We shall perform a simple reduction so that  $s$  is inside this component  $C$ , has only one outgoing arc, and every vertex has degree three as follows. First we collapse the maximal component containing  $s$  inside  $f_s$  into a vertex. To ensure the uniqueness of right-most paths starting from  $s$ , we create a new vertex  $s'$  pointing to  $s$ , and finally, we add additional vertices to ensure the degree constraint (more precisely, we replace a degree  $d$  vertex by a binary tree on  $d$  leaves). See Figure 7.

### 6.1 Overview

We decompose the problem of checking usefulness of component  $C$  into many subproblems and will deal with each subproblem (almost) independently. The basis of our decomposition is in slicing the component with the *lowest-floor path*  $F^*$  and the *top-most ceiling*  $U^*$ . These two paths then divide the component  $C$  into two parts: (1) the primary strip and (2) the open strip. Figure 8 illustrates

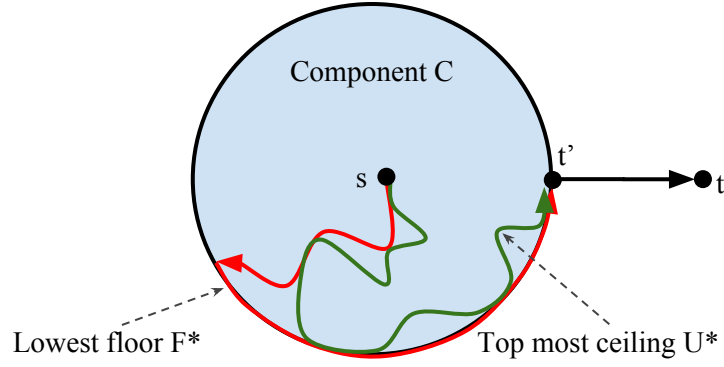


Figure 8: the lowest-floor path  $F^*$  and the top-most ceiling  $U^*$  in the inside case.

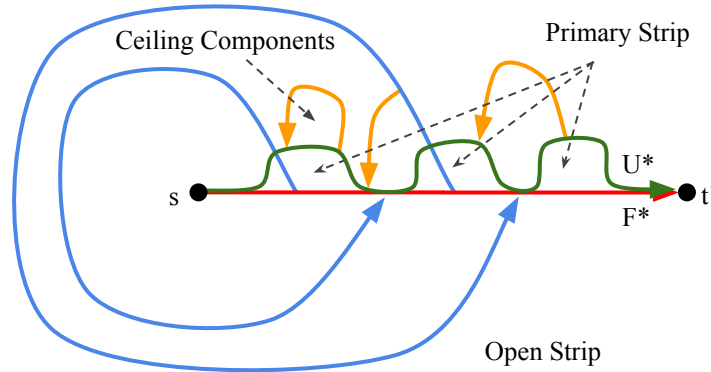


Figure 9: The types of major strips in the inside case.

the paths  $F^*$  and  $U^*$ , and Figure 9 shows the schematic view of the primary and the open strips. All types of strips and components will be defined next.

### 6.1.1 The lowest-floor path $F^*$ .

We mainly apply the algorithm from Section 5 to the primary strip defined by two paths  $F^*$  and  $U^*$ , which may not be arc-disjoint. The path  $F^*$  is the *lowest-floor path* defined similar to the outside case: We compute  $F^*$  by finding the right-most path  $P$  from  $s$  to  $t$  (hence,  $P$  is an  $s, t$ -path). While  $t$  maybe outside of the component  $C$ , the right-most path  $P$  goes through some exit  $t'$  on the boundary of  $C$ . We let  $P'$  be the subset of  $P$  inside  $C$ . We then extend  $P'$  along the boundary of  $C$  to include all the exits to obtain  $F^*$  (see Figure 10).

### 6.1.2 Flipped paths.

Unlike the outside case, the path  $F^*$  in the inside case does not divide the component  $C$  into two parts because the source vertex  $s$  is inside some inner face of  $C$ . This leaves us more possibilities of paths that we have not encountered in the outside case, called “flipped paths”.

To be precise, we say that an arc  $uu'$  leaves  $F^*$  to the right if  $u \in V(F^*)$  and  $uu'$  lies on the right of  $F^*$ . A path  $P$  leaves  $F^*$  to the right if its first arc of  $P$  leaves  $F^*$  to the right. We can

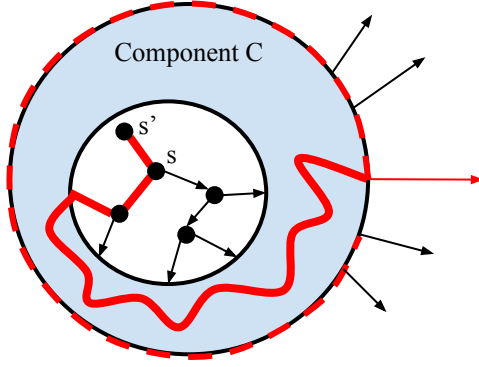


Figure 10: How to find  $F^*$ : path  $P$  is shown in solid red. The extension on the boundary is shown as a dashed line.

similarly say that an arc  $uu'$  enters  $F^*$  from the right, if  $u' \in V(F^*)$  and  $uu'$  lies on the right of  $F^*$ , and a path  $P$  enters  $F^*$  from the right if its last arc enters  $F^*$  from the right. We can define similarly for the case when a path leaves or enters  $F^*$  from the left. We also use the same notion for path leaving (resp., entering)  $U^*$ .

Consider a simple path  $P$  which is internally vertex disjoint from  $F^*$  but whose end vertices are in  $V(F^*)$ . Let  $u$  and  $v$  be the start and end vertices of  $P$ , respectively. We say that  $P$  is *flipped* if it leaves  $F^*$  to the left and enters  $F^*$  from the right. Note that since  $F^*$  is right-most, no path leaves  $F^*$  from the right and then enters  $F^*$  again from the right.

It is also impossible that a path leaves  $F^*$  to the right and enters  $F^*$  from the left as either case it contradicts the fact that  $F^*$  is right-most or it introduces a clockwise cycle.

If  $P$  leaves  $F^*$  to the left and enters  $F^*$  from the left,  $P$  is not flipped. In this case, we can use the same definition from the previous section. Recall that we can define an order  $\phi$  of vertices in  $V(F^*)$ . Thus, with respect to  $F^*$  and the ordering  $\phi$ , the path  $P$  starting at  $u$  and ending at  $v$  can either be forward or backward depending on  $\phi(u)$  and  $\phi(v)$ .

If  $P$  is flipped and  $\phi(v) > \phi(u)$ , we say that  $P$  is a *forward flipped path*. On the other hand,  $P$  is a *backward flipped path* if  $P$  is flipped and  $\phi(v) \leq \phi(u)$ . For brevity, we refer to forward non-flipped and backward non-flipped paths as only forward and backward paths, respectively. See Figure 11 for an illustration.

### 6.1.3 Primary Strip.

The *primary strip* is defined by two paths, which may not be arc-disjoint  $F^*$  and  $U^*$ . The path  $F^*$  is the lowest-floor path. The path  $U^*$  is the *top-most ceiling*, which is an  $s, t$ -path such that the strip  $C_{U^*, F^*}$  encloses all (non-flipped) forward paths (w.r.t.  $F^*$ ).  $U^*$  is essentially the left-most path from  $s$  to  $t$ . Section 6.2 describes how we find  $U^*$  given  $F^*$

### 6.1.4 Open Strip.

The *open strip*  $\hat{C}$  is defined to be everything outside the primary strip, i.e.,  $\hat{C} := C - C_{U^*, F^*}$ . See Figure 11 for illustrations.

To deal with arcs in  $\hat{C}$ , we first characterize the first set of useless arcs: ceiling arcs. An arc

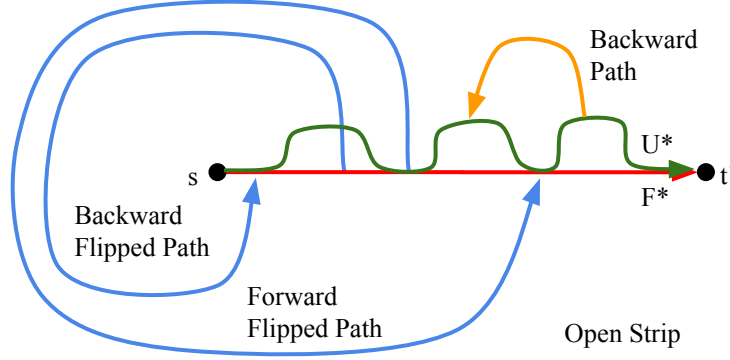


Figure 11: The types of paths in the open strip.

$e \in \hat{C} = (u, v)$  is a *ceiling arc* if  $u$  can be reached only from paths leaving  $F^*$  to the left and  $v$  can reach  $t$  only through paths entering  $F^*$  from the left as well. These arcs form “ceiling components”. (Note that paths that form ceiling components are all backward paths because all the forward paths (w.r.t.  $F^*$ ) have been enclosed in  $C_{U^*, F^*}$ .) We shall prove that these paths are useless in Lemma 22. Therefore, as a preprocessing step, after they are found, we delete them all from  $\hat{C}$ .

To deal with remaining arcs in  $\hat{C}$ , we first find strongly connected components  $H_1, H_2, \dots$  in  $\hat{C}$ . Arcs outside strongly connected components then form a directed acyclic graph  $\hat{D}$  that represents essentially the structures of all flipped paths. We can process each component  $H_i$  using the outside algorithm only if we know which adjacent arcs in  $\hat{D}$  are valid entrances and exits. For arcs in  $\hat{D}$  that belong to some flipped path, we can use a dynamic programming algorithm to compute all reachability information needed to determine their usefulness.

Our algorithm can be described shortly as follows.

- **Initialization.** We compute the lowest-floor path  $F^*$  and the top-most ceiling  $U^*$ , thus forming the primary strip  $C_{U^*, F^*}$  and the open strip  $\hat{C}$ .
- **The open strip (1): structures of flipped paths.** We find all strongly connected components in  $\hat{C}$  and collapse them to produce a directed acyclic graph (DAG)  $\hat{D}$ . We compute the reachability information of arcs in  $\hat{D}$  and use it to determine the usefulness of arcs in  $\hat{D}$ .
- **The open strip (2): process each strongly connected component.** Then we apply the outside algorithm to each strongly connected components in  $\hat{C}$ .
- **The primary strip.** We determine the usefulness of arcs in the primary strip.

In the following subsections, we describe how to deal with each part of  $C$ .

## 6.2 Initialization: Finding the Primary and Open Strips

In this section, we discuss the initialization step of our algorithm, which computes the primary and the open strips. In this step, we also compute the reachability information of flipped paths, which will be used in the later part of our algorithm.

As discussed in the previous section, we compute the lowest-floor path  $F^*$  by finding the right-most  $s, t$ -path, which is unique and well-defined because  $s$  has a single arc leaving it. The path  $F^*$  is contained in  $C$  because we assume that  $t$  is an exit vertex on the boundary of  $C$ .

The top-most ceiling  $U^*$  can be computed by simply computing the left-most path from  $s$  to  $t$  and choosing such path that enters  $F^*$  from the left (this is to guarantee that  $U^*$  is not a flipped path). This again can be done in linear-time by a standard implementation of depth-first-search.

Now we have the primary strip  $C_{U^*, F^*}$ , which we claim that it encloses all forward paths (w.r.t.  $F^*$ ).

**Lemma 16.** *Every forward path w.r.t.  $F^*$  is enclosed in  $C_{U^*, F^*}$ .*

*Proof.* Suppose to a contrary that there exists a forward path  $P$  not enclosed by  $C_{U^*, F^*}$ . Then  $P$  must intersect with  $U^*$  or  $F^*$ . If it is the former case, then  $P$  must have a subpath  $Q$  that leaves and enters  $U^*$  on the left (this is because  $U^*$  is on the left of  $F^*$ ). We choose a subpath  $Q$  in which the start vertex of  $Q$ , say  $u$ , appears before its end vertex, say  $v$ , on  $U^*$ . Such path  $Q$  must exist because, otherwise,  $P$  would have a self-intersection inside  $C_{U^*, F^*}$ . We may further assume that  $Q$  is a minimal such path, which means that arcs of  $Q$  lie entirely on the left of  $U^*$ . But, then  $U^*$  would not be a left-most path because the search algorithm would have followed  $Q$  instead of the  $u, v$ -subpath of  $U^*$ , a contradiction.

The case that  $P$  intersects  $F^*$  is similar, and such path would contradict the fact that  $F^*$  is the right-most path. Therefore, all the forward paths (w.r.t.  $F^*$ ) must be enclosed in  $C_{U^*, F^*}$ .  $\square$

Next consider the remaining parts of the component, which forms the open strip  $\hat{C} = C - C_{U^*, F^*}$ . We have as a corollary of Lemma 16 that there is no simple  $s, t$ -path in the open strip that leaves  $F^*$  from the right.

**Corollary 17.** *No simple  $s, t'$ -path that lies on the right of  $F^*$ .*

*Proof.* Suppose there is an  $s, t$ -path  $P$  in the open strip that leaves  $F^*$  from the right. Then we have three cases. First, if  $P$  also enters  $F^*$  from the right, then  $P$  cannot be a forward-path and cannot have any forward subpath by Lemma 16. Thus,  $P$  must have a self-intersection, contradicting to the fact that  $P$  is a simple path. Second, if  $P$  enters  $F^*$  from the left, then we can find a  $u, v$ -subpath  $Q$  of  $P$  such that  $u$  appears before  $v$  in  $F^*$  and  $Q$  is arc-disjoint from  $F^*$ . But, then the right-first-search algorithm would have followed  $Q$  instead of the  $u, v$ -subpath of  $F^*$ , a contradiction. Otherwise, if  $P$  never enters  $F^*$ , then it must go directly to  $t$ . But, then the right-first-search algorithm would have followed the subpath of  $P$  that leaves  $F^*$ , a contradiction.  $\square$

### 6.3 Working on the Open Strip (1): Dealing with Arcs in Flipped Paths

To check if an arc in a DAG  $D$  formed by contracting strongly connected components  $H_1, H_2, \dots$  in the open strip is useful, it suffices to check if it is contained in some useful flipped path w.r.t.  $F^*$ .

We first prove characterization of useful flipped paths. Section 6.3.1 describe how we can check if an arc belongs to any useful paths according to the characterization based on reachability information.

We state our first observation that forward flipped paths are useful.

**Lemma 18.** *Every forward flipped path w.r.t.  $F^*$  is useful.*



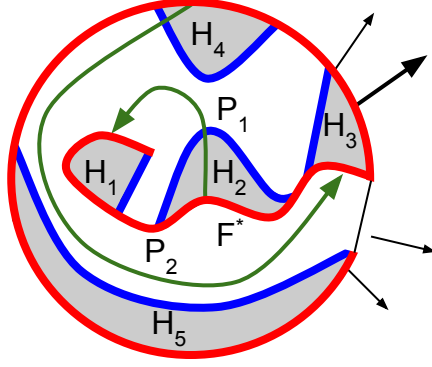


Figure 12: An example of flipped backward paths  $P_1$  (useless) and  $P_2$  (useful).  $F^*$  is shown in red and  $U^*$  is shown in blue. There are 5 humps (shown in gray).

*Proof.* Consider any forward flipped path  $P$ . By the definition of the forward flipped path,  $P$  shares no arcs with  $F^*$ . Moreover,  $P$  starts at some vertex  $v \in F^*$  and ends at some vertex  $w \in F^*$  such that  $v$  appears before  $w$  in  $F^*$ . Note that  $F^*$  is a path that goes from  $s$  to the vertex  $t'$  in the boundary of the component  $C$ . We extend  $F^*$  to a simple  $s, t$ -path  $R$  by adding a  $t', t$ -path that is not in the component  $C$ . We then replace the  $u, v$ -subpath of  $R$  by  $P$ , thus getting a new path  $R'$ , which is a simple path because  $R$  shares no arcs with  $P$  and contains no vertices of  $V(P) - \{v, w\}$ . Thus,  $R'$  is a simple  $s, t$ -path, implying that  $P$  is useful.  $\square$

If an arc is not contained in any forward flipped path, it must be in some backward flipped path. In most cases, backward flipped paths are useless except when there exists an exit that provides alternative route to  $t$ .

Consider the primary strip enclosed by the lowest-floor  $F^*$  and top-most ceiling  $U^*$ . If we remove arcs that  $F^*$  and  $U^*$  share, i.e.,  $F^* \cap U^*$ , the strip can be decomposed into many subgraphs, which we refer to as humps. Formally, a *hump* is a strip obtained from the primary strip by removing  $F^* \cap U^*$ . We can also order humps by the ordering of their first floor vertices. The structure of humps is important in determining the usefulness of a flipped path. See Figure 12.

The lemma below shows the conditions when arcs in a backward flipped path can be useful.

**Lemma 19.** *Let  $P$  be any backward flipped-path such that  $P$  starts at a vertex  $u \in F^*$  and ends at a vertex  $v \in F^*$ . Then  $\hat{P} = P \cap \hat{C}$  (the subpath of  $P$  in the open strip) is useful iff  $P$  ends at some hump  $H$  (thus,  $H$  contains  $v$ ) such that either*

1. both  $u$  and  $v$  are in the same hump  $H$ , and  $v$  is not the first vertex in  $H$ .
2.  $u$  is not in the hump  $H$ , and there is an exit after  $v$  in the hump  $H$ .

*Proof.* First, we show that if the end vertex  $v$  of  $P$  is not in any hump, then  $\hat{P} = P \cap \hat{C}$  is useless. We assume wlog that the start vertex  $u$  of  $P$  is also not contained in any hump; otherwise, we contract all vertices in the hump (that contains  $u$ ) to the vertex  $u$ . Thus,  $P = \hat{P}$ . It suffices to show that there is no  $s, u$ -path  $J$  that is vertex disjoint from  $P$ , which will imply that  $P$  cannot be a useful path. To see this, assume a contradiction that there is an  $s, u$ -path  $J$  that is vertex disjoint from  $P$ . Then  $J$  has to leave  $F^*$  at some vertex  $x$  and then enters  $F^*$  again at some vertex

$y$  (it is possible that  $y = u$ ) in such a way that  $x, y, u, v$  appears in the order  $(x, v, y, u)$  on  $F^*$ . Let  $J'$  be the  $x, y$ -subpath of  $J$ . If  $J'$  is contained in the primary strip, then  $J'$  would induce a hump containing  $v$ , a contradiction. So, we assume that  $J'$  is not contained in the primary strip. If  $J'$  leaves  $F^*$  from the right, then  $J'$  cannot enter  $F^*$ ; otherwise, it would contradict the fact that  $F^*$  is the right-most path. Hence, we are left with the case that  $J'$  leaves  $F^*$  from the left and then enters again from the right. Since  $J$  and  $P$  are vertex-disjoint (and so is  $J'$ ),  $J'$  cannot enter  $F^*$  at any vertex that appears after  $v$  (including  $u$  and  $y$ ) because, otherwise,  $J'$  has to cross the path  $P$ . Thus, we again have a contradiction. Consequently, since there is no path from  $s$  that can reach  $u$  or the hump containing  $u$  without intersecting with  $\hat{P}$ ,  $\hat{P}$  cannot be a useful path.

Next consider the case that  $v$  is contained in some hump  $H$ . Let  $w$  be a vertex in  $P \cap U^*$ , i.e.,  $w$  be a vertex in the ceiling intersecting with  $P$ . Also, let  $\hat{P}$  be  $P \cap \hat{C}$ .

For the backward direction, we construct a useful path  $Q$  containing  $\hat{P}$  by taking the union of the prefix of the top-most ceiling  $U^*$  up to  $w$ ,  $\hat{P}$  and the subpath  $P_F$  of the floor  $F^*$  after  $v$ . In Case (1),  $P_F$  is the suffix of  $F^*$  after  $v$ , and in Case (2),  $P_F$  starts from  $v$  and ends at the exit. It can be seen that  $Q$  is a simple path from  $s$  to an exit, meaning that  $Q$  is a useful path and so is  $\hat{P}$ .

Let's consider the forward direction. We prove by contraposition. There are two subcases we have to consider: (i)  $u$  and  $v$  are in the same hump  $H$ , but  $v$  is the first vertex in the hump and (ii)  $u$  is not in the hump  $H$  and there is no exit after  $v$  in  $H$ .

In Subcase (i), any useful path  $Q$  containing  $\hat{P}$  whose prefix up until  $\hat{P}$  is entirely in the primary strip has to go through  $v$ ; therefore, it cannot be simple. Consider the case when there is a useful path  $Q$  containing  $\hat{P}$  outside the primary strip. The prefix  $Q'$  of this path must reach  $w$  without touching  $v$  (because  $v \in \hat{P}$ ). Observe that the union of  $H$  and  $\hat{P}$  contains a cycle enclosing the source  $s$ , and by the construction of  $U^*$ , there is no path from  $s$  entering  $H \cap U^*$  (otherwise, it would have been included to  $U^*$ ). Consequently, because of planarity, the prefix  $Q'$  must cross  $\hat{P}$ , meaning that  $Q$  cannot be simple.

Now we deal with Subcase (ii). Let  $x$  be the last floor vertex of  $H$ . First note that, by the choice of  $F^*$ , any path from  $s$  to  $w$  not intersecting  $\hat{P}$  cannot cross  $F^*$  to the right; therefore, it has to use  $x$ . Because of the same reason, any path from  $v \in F^*$  to  $t$  cannot leave  $F^*$  to the right; thus, it must also go through  $x$  as well. Thus, a path from  $s$  to  $t$  containing  $\hat{P}$  in this case cannot be simple.  $\square$

### 6.3.1 Reachability information

After we compute the primary strip and the open strip, and collapse strongly connected components in  $\hat{C}$ , we run a linear-time preprocessing to compute reachability information.

For each vertex  $u$  in  $\hat{D}$  we would like to compute  $\text{first}(u) \in F^*$  defined to be the first vertex  $w$  in  $F^*$  such that there is a path from  $s$  to  $u$  that uses  $w$  but not other vertex in  $F^*$  after  $w$ . We also want to compute  $\text{last}(u) \in F^*$  defined to be the last vertex  $w'$  in  $F^*$  such that there is a path from  $u$  to  $t$  that uses  $w'$  as the first vertex in  $F^*$ .

To find  $\text{first}(u)$ , we perform left-first search tree  $T$  from  $s$ . We set  $\text{first}(u)$  for  $u \in \hat{C}$  to be its closest ancestor in the floor  $F^*$ . The mapping  $\text{last}(u)$  can be done similarly but going backward from  $t$ .

### 6.3.2 Checking arcs in $\hat{D}$

Now we describe our linear-time algorithm for determining the usefulness of arcs in  $\hat{D}$

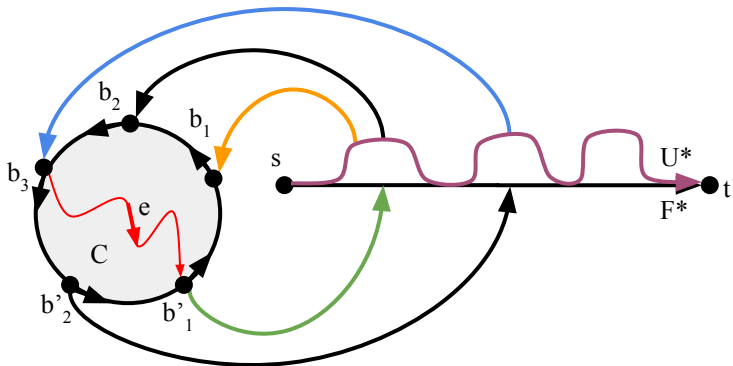


Figure 13: The arc  $e$  in the red path is useful when considering  $C$  as an outside case instance. However, using the blue path as an entering path (at the entrance  $b_3$ ) and the green path as a leaving path (at the exit  $b'_2$ ) is not enough to show that. One can, instead, use the orange path (at the entrance  $b_1$ ) as an entering path together with parts of the boundary to construct a forward flipped path that contains  $e$ .

Consider arc  $e = (u, v) \in \hat{D}$ . If  $\phi(\text{first}(u)) > \phi(\text{last}(v))$ , then there exist a flipped forward path containing  $e$ . Thus, by Lemma 18,  $e$  is useful.

If  $\phi(\text{first}(u)) \leq \phi(\text{last}(v))$ , we need to check conditions in lemma 19. To do so, we need to maintain additional information on every vertex  $v$  in  $F^*$ , namely the hump  $H(v)$  that contains it, a flag representing if  $v$  is the first vertex in the hump, and a flag representing if the hump  $H(v)$  contains a exit. Since these data can be preprocessed in linear-time, we can perform the check for each arc  $e$  in constant time.

## 6.4 Working on the Open Strip (2): Dealing with Arcs in Strongly Connected Components

Consider a strongly connected component  $H_i$  in the open strip. If no arcs going into  $H_i$  are useful or no arcs leaving  $C$  are useful, clearly every arc in  $H_i$  are useless. However, it is not obvious that applying the previous outside algorithm simply works because entering and leaving arcs determine the usefulness of the path. See Figure 13, for example.

The following lemma proves that we can take the strongly connected component  $H_i$  and build an outside case instance by attaching, for each useful arc in the DAG entering  $H_i$  or leaving  $H_i$ , corresponding to an entrance or an exit.

**Lemma 20.** *Every arc in an outside instance of  $H_i$  is useful iff it is useful in the original graph.*

*Proof.* The backward direction is straight-forward. Consider the forward direction: we would like to show that if an arc  $e$  is useful in the instance, there exists a useful path containing it.

Suppose that  $H_i$  has  $n_1$  entrances and  $n_2$  exits. From Lemma 3 and an observation inside Lemma 14 that entrance-exit pairs do not appear interleaving, we know that  $H_i$  is enclosed by a counter-clockwise cycle and we can name all entrances as  $b_1, b_2, \dots, b_{n_1}$  and all exits as  $b'_1, b'_2, \dots, b'_{n_2}$  in such a way that they appear in counter-clockwise order as

$$b_1, b_2, \dots, b_{n_1}, b'_1, b'_2, \dots, b'_{n_2}.$$

Consider an arc  $e$  which is useful in this instance. Since  $e$  is useful, there is a vertex-disjoint path  $P$  from some entrance  $b_i$  to some exit  $b'_j$  containing  $e$ . Since the boundary arcs form a counter-clockwise cycle, we can extend  $P$  to start from  $b_1$  by adding boundary arcs from  $b_1$  to  $b_i$ . We can also extend  $P$  to reach  $b'_{n_2}$  by adding arcs from  $b'_j$  to  $b'_{n_2}$ .

Since an arc entering  $C$  to  $b_1$  is useful and an arc leaving  $C$  from  $b'_{n_2}$  is also useful, we can construct a useful path containing  $e$  by joining a path from  $s$  to  $b_1$ , the path  $P$ , and a path from  $b'_{n_2}$  to  $t$ . Thus,  $e$  is useful.  $\square$

## 6.5 Working on the Primary Strip

Given the primary strip  $C_{U^*,F^*}$ , we can use the algorithm from Section 5 to find all useful arcs inside  $C_{U^*,F^*}$ . The next lemma proves the correctness of this step.

**Lemma 21.** *A useful arc  $e$  in  $C$  is useful iff it is useful in  $C_{U^*,F^*}$ .*

*Proof.* The backward direction is obvious. We focus on the forward direction. Assume for contradiction that there exist a useful arc  $e$  in  $C$  but  $e$  is not useful in  $C_{U^*,F^*}$ . Consider a useful path  $P$  from  $s$  to  $t$  containing  $e$ . Since  $e$  is inside  $C_{U^*,F^*}$ ,  $P$  must cross the lowest-floor  $F^*$  or the top-most ceiling  $U^*$ . However,  $P$  cannot cross  $U^*$  as it would create a forward path outside the primary strip (contradicting Lemma 16). Now suppose that  $P$  crosses  $F^*$  at some vertex  $w$  before  $e$ . If  $P$  do not leave the primary strip after  $w$ , clearly  $e$  must be useful inside the primary strip. If  $P$  leaves the primary strip at the ceiling after  $e$ , the subpath of  $P$  containing  $e$  is again a forward path. We also note that  $P$  cannot leave  $C_{U^*,F^*}$  at the floor because it would cross itself. Since we reach contradiction in every case, the lemma is proved.  $\square$

## 6.6 The Ceiling Components are Useless

In this section, we prove that ceiling components are useless. A ceiling components are formed from arcs  $e = (u, v)$  in  $\hat{C}$  which can be reachable only from paths leaving  $F^*$  to the left and  $v$  can reach  $t$  only through paths entering  $F^*$  from the left.

**Lemma 22.** *Every ceiling arc  $e$  is useless.*

*Proof.* Note that  $e$  does not lie in a forward path; otherwise  $e$  would be in the primary strip.

Let  $P$  be a useful path from  $s$  to  $t$  containing  $e$ . Let  $u'$  be the last vertices of  $P \cap U^*$  before reaching  $e$  and  $v'$  be the first vertices of  $P \cap U^*$  after leaving  $e$ .

Because of the degree constraint,  $P'$  must use the only incoming arc of  $u'$ , defined to be  $e_u$  and the only outgoing arc of  $v'$ , defined to be  $e_v$ . Let  $P_1$  be the prefix of  $P'$  from  $s$  to the head of  $e_u$  and  $P_2$  be the suffix of  $P'$  from tail of  $e_v$  to  $t$ . The only way  $P_2$  can avoid crossing with  $P_1$  is to cross  $U^*$ ; however, this creates a clockwise cycle.  $\square$

## 7 Conclusions and open problems

We discuss briefly the open problems. In this paper we assume that each vertex has degree three and the graph contains no clockwise cycles. Without the degree constraint, the structure of useful paths becomes much more involved. Specifically, inner arcs in a strip forming backward paths can be useful as shown in Figure 14. This also creates a lot of dependencies between strips inside the strip decomposition.

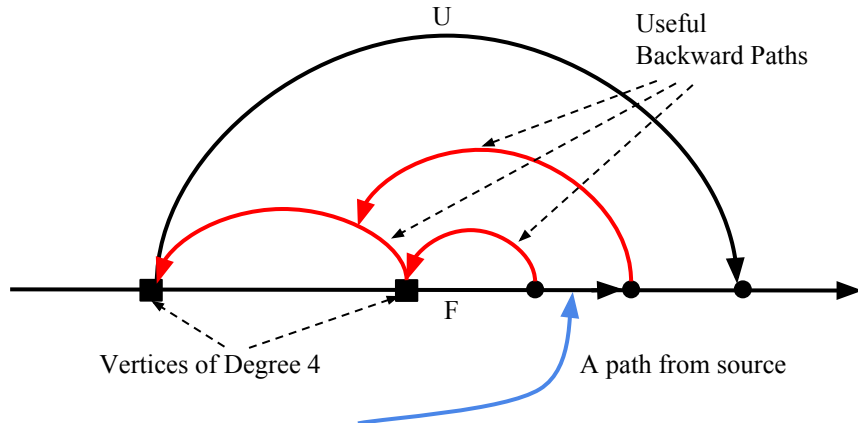


Figure 14: A strip  $C_{U,F}$  is shown with inner useful arcs (shown in red). They are useful because of a dotted path reaching an entrances at the floor path. Vertices of degree four are shown as squares.

An obvious open problem is to remove these assumptions and also to devise a linear-time algorithm. Another interesting one would be to consider a larger class of graphs on surfaces.

**Acknowledgement.** We thank Joseph Cheriyan for introducing us the flow network simplification problem. We also thank Karthik C.S. for pointing out some typos.

## References

- [BBV00] Therese C. Biedl, Brona Brejová, and Tomáš Vinar. Simplifying flow networks. In *Mathematical Foundations of Computer Science 2000, 25th International Symposium, MFCS 2000, Bratislava, Slovakia, August 28 - September 1, 2000, Proceedings*, pages 192–201, 2000. [2](#)
- [BH13] Glencora Borradaile and Anna Harutyunyan. *Maximum  $st$ -Flow in Directed Planar Graphs via Shortest Paths*, pages 423–427. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013. [3](#)
- [BK09] Glencora Borradaile and Philip N. Klein. An  $O(n \log n)$  algorithm for maximum  $st$ -flow in a directed planar graph. *J. ACM*, 56(2), 2009. Preliminary version in SODA’06. [1](#), [2](#), [3](#)
- [BKM<sup>+</sup>11] G. Borradaile, P. N. Klein, S. Mozes, Y. Nussbaum, and C. Wulff-Nilsen. Multiple-source multiple-sink maximum flow in directed planar graphs in near-linear time. In *2011 IEEE 52nd Annual Symposium on Foundations of Computer Science*, pages 170–179, Oct 2011. [3](#)
- [EK13] David Eisenstat and Philip N. Klein. Linear-time algorithms for max flow and multiple-source shortest paths in unit-weight planar graphs. In *Proceedings of the Forty-fifth Annual ACM Symposium on Theory of Computing, STOC ’13*, pages 735–744, New York, NY, USA, 2013. ACM. [3](#)

- [Eri10] Jeff Erickson. Maximum flows and parametric shortest paths in planar graphs. In *Proceedings of the Twenty-first Annual ACM-SIAM Symposium on Discrete Algorithms, SODA '10*, pages 794–804, Philadelphia, PA, USA, 2010. Society for Industrial and Applied Mathematics. [3](#)
- [FF56] Lester R Ford and Delbert R Fulkerson. Maximal flow through a network. *Canadian journal of Mathematics*, 8(3):399–404, 1956. [1](#)
- [FHW80] Steven Fortune, John E. Hopcroft, and James Wyllie. The directed subgraph homeomorphism problem. *Theor. Comput. Sci.*, 10:111–121, 1980. [2](#)
- [Fre87] Greg N. Frederickson. Fast algorithms for shortest paths in planar graphs, with applications. *SIAM J. Comput.*, 16(6):1004–1022, 1987. [1](#)
- [Has81] Refael Hassin. Maximum flow in (s, t) planar networks. *Inf. Process. Lett.*, 13(3):107, 1981. [1](#)
- [HJ85] Refael Hassin and Donald B. Johnson. An  $o(n \log^2 n)$  algorithm for maximum flow in undirected planar networks. *SIAM J. Comput.*, 14(3):612–624, 1985. [1](#)
- [HKRS97] Monika Rauch Henzinger, Philip N. Klein, Satish Rao, and Sairam Subramanian. Faster shortest-path algorithms for planar graphs. *J. Comput. Syst. Sci.*, 55(1):3–23, 1997. Preliminary version in STOC'94. [1](#)
- [INSW11] Giuseppe F. Italiano, Yahav Nussbaum, Piotr Sankowski, and Christian Wulff-Nilsen. Improved algorithms for min cut and max flow in undirected planar graphs. In *Proceedings of the 43rd ACM Symposium on Theory of Computing, STOC 2011, San Jose, CA, USA, 6-8 June 2011*, pages 313–322, 2011. [1](#)
- [IS79] Alon Itai and Yossi Shiloach. Maximum flow in planar networks. *SIAM J. Comput.*, 8(2):135–150, 1979. [1](#)
- [KNK93] Samir Khuller, Joseph Naor, and Philip N. Klein. The lattice structure of flow in planar graphs. *SIAM J. Discrete Math.*, 6(3):477–490, 1993. [2](#)
- [KRHS94] Philip N. Klein, Satish Rao, Monika Rauch Henzinger, and Sairam Subramanian. Faster shortest-path algorithms for planar graphs. In *Proceedings of the Twenty-Sixth Annual ACM Symposium on Theory of Computing, 23-25 May 1994, Montréal, Québec, Canada*, pages 27–37, 1994. [1](#)
- [MN95] Gary L. Miller and Joseph (Seffi) Naor. Flow in planar graphs with multiple sources and sinks. *SIAM Journal on Computing*, 24(5):1002–1017, 1995. [3](#)
- [Rei83] John H. Reif. Minimum s-t cut of a planar undirected network in  $o(n \log^2(n))$  time. *SIAM J. Comput.*, 12(1):71–81, 1983. [1](#)
- [Wei97] Karsten Weihe. Maximum (s, t)-flows in planar networks in  $o(|v| \log |v|)$ -time. *J. Comput. Syst. Sci.*, 55(3):454–476, 1997. Preliminary version in FOCS'94. [2](#)