On the Complexity of 2D Discrete Fixed Point Problem, X. Chen and X. Deng, ICALP'06.

Definition. TFNP Let $R \subseteq \{0,1\}^* \times \{0,1\}^*$ be a relation such that:

(a) Given x, y, it can be checked in polynomial time if $(x, y) \in R$.

(b) There is a polynomial p such that if $(x, y) \in R$ then $|y| \leq p(|x|)$.

(c) For every x there exists a y with $(x, y) \in R$.

The total NP search problem Q_R is this: given an input x, return a y such that $(x, y) \in R$. We use TFNP to denote the class of total NP search problems.

Definition (reduction)**.** An NP search problem Q_A is polynomial-time reducible to problem Q_B if there exists a pair (f, g) of polynomial-time computable functions such that, for every input x of Q_A, if y satisfies $(f(x), y) \in B$, then $(x, g(y)) \in A$.

Definition (LEAFD, a.k.a. End-Of-The-Line)**.** The input is a pair $(M, 0^k)$, where M is a polynomial-time Turing machine generating a directed graph G with vertex set $\{0,1\}^k$ and $\Delta^+(G) = \Delta^-(G) = 1$: For any $v \in \{0,1\}^k$, $M(v)$ gives the unique in-neighbour and out-neighbour of v if they exist. There is an arc from 0^k to 1^k. The output is a directed leaf other than 0^k, i.e. a vertex with in-degree + out-degree = 1.

PPAD was defined by Papadimitriou in 1990:

Definition (Polynomial Parity Arguments on Directed graphs (PPAD))**.** PPAD is the set of total NP search problems that are polynomial-time reducible to LEAFD. From its definition, LEAFD is complete for PPAD.

Definition (The graph G_n and C_n)**.** G_n is the following "grid embedding" of K_n. Refer to Fig. 6 of the handout. G_n embeds in a $3n^2 \times 6n$ grid. C_n is the set of subgraphs H of G_n with $\Delta^+(H) = \Delta^-(H) = 1$ and $d^-(0, 0) = 0$.

Definition (RLEAFD)**.** The input is a (Turing Machine description of a) graph in C_{2^k}. The output is a directed leaf other than $(0, 0)$.

Definition ($C(G)$)**.** For an arbitrary graph G, $C(G) \in C_n$ is the following "grid pseudo-embedding" of G. Refer to Fig. 7 of the handout.

Note that $C(G)$ is not really an embedding of G. However, it preserves the leaves of G.

Lemma 1. *LEAFD can be reduced to RLEAFD. Therefore, RLEAFD is PPAD-complete.*

Proof. Let $(M, 0^k)$ be an input instance of LEAFD and G be the directed graph specified by M. It is tedious, but not hard, to show that one can build a Turing machine K from M in polynomial time, such that As an input of RLEAFD, the graph generated by K is exactly $C(G) \in C_{2^k}$. Moreover, given any directed leaf of $C(G)$, we can locate a directed leaf of G easily. In this way, we get a reduction from LEAFD to RLEAFD and the lemma follows. $\square$

Definition (T_n, 2D-SPERNER)**.** The input is a pair $(F, 0^k)$ where F is a polynomial-time Turing machine which produces a Sperner 3-colouring c on T_{2^k}. Here $c(p) = F(p) \in \{0, 1, 2\}$ for every vertex p of T_{2^k}. The output is a trichromatic triangle (which exists by Sperner's lemma).

Theorem 2. *Search problem 2D-SPERNER is PPAD-complete.*

Proof. First we show that $RLEAFD \leq 2D - SPERNER$. From an input $(K, 0^k)$ of RLEAFD (which is a directed graph G embedded in a $3.2^{2k} \times 6.2^k$ grid, see Fig. 8 (right)), it is possible to build a Turing machine M_K that gives a Sperner 3-colouring of $T_{2^{2k+5}}$ (see Fig. 9). From a multicoloured triangle of this triangulation, one can find a directed leaf of G.

(u_1, u_2) is mapped to $(3u_1 + 3, 3u_2 + 3)$.

- vertices on an edge are coloured black (except the two endpoints, which are white).
- vertices just on right-side of an edge are coloured white.
- vertices just on left-side of an edge are coloured gray.
- other vertices are white.
- There are exceptions around (3,3) to avoid a multicoloured triangle around the trivial directed leaf (0,0).
- There are exceptions in the first column to make it a Sperner's colouring (cannot use white in first column).

Second, we show that $2D - SPERNER \leq LEAFD$. □

Problem	Category	Reference
3D Sperner	Fixed Point Theorems	Papadimitriou'94
2D Sperner, 2D Brouwer	Fixed Point Theorems	Chen, Deng'06-sperner
3D Brouwer	Fixed Point Theorems	Daskalakis, Goldberg, Papadimitriou '05
2-Nash	Nash	Chen, Deng'06
Scarf's Lemma	Combinatorics	Kintali, Poplawski, Rajaraman, Sundaram, Teng '09
2D-Tucker	Topology	Palvolgyi'09