Cops and a Fast Robber on Bounded-Degree and Random Graphs

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UBC and SFU

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joint work with Noga Alon

Remarks

ANIMATE!

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1. perfect-information game
2. More than one cops can be at the same vertex.
3. Robber cannot jump over a cop.
4. Moves are deterministic.
5. When describing a strategy for the cops, we assume the robber is clever; and vice versa.
6. Interested in minimum number of cops to guarantee capture.

What's known

- ✓ On a path/complete graph one cop suffices.
- ✓ On a cycle/grid, two cops suffice (bus problem).
- ✓ On a planar graph, three cops suffice. [Aigner,Fromme'84]
- ✓ Meyniel conjectured $L\sqrt{n}$ cops suffice for any graph. [Frankl'87]
  We don't have a proof that $n^{0.99}$ cops suffice for all graphs!
  n : number of vertices
- ✓ On a random graph $L\sqrt{n}$ cops suffice with high prob. [Prałat,Wormald'15]

The fast robber variant

ANIMATE!

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ANIMATE!

Definition (The Game of Cops and Robber)

- ✓ In the beginning,
  - First, each cop chooses a starting vertex.
  - Then, the robber chooses a starting vertex.
- ✓ In each round,
  - First, each cop chooses to stay or go to an adjacent vertex.
  - Then, the robber chooses to stay, or move along a cop-free path.
- ✓ The cops capture the robber if, at some moment, a cop is at the same vertex with the robber.
- ✓ Cop number of $G = c(G)$

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- ✓ On a path/tree/complete graph one cop suffices.
- ✓ On a cycle two cops suffice.
- ✓ On an $m \times m$ grid, m cops are necessary and sufficient (bus problem).
- ✓ Computing $c(G)$ is NP-hard.
  [Fomin, Golovach, Kratochvíl'08]
- ✓ For every n, there exists a graph with $c(G) = \Theta(n)$.
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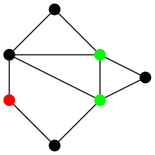
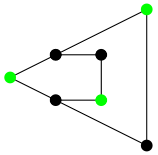
Today we study cop numbers of bounded-degree and random graphs.

Dominating Set

$N(S) :=$ (closed) neighbourhood of set S
A is dominating set : $N(A) = V(G)$

Example





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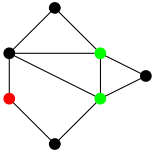
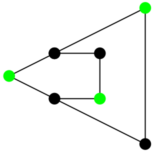


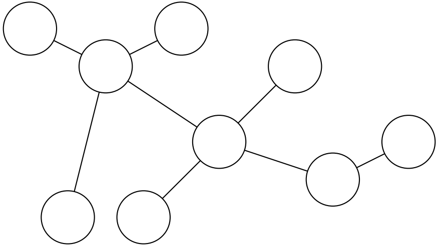


$c(G) \leq \gamma(G) =$ size of a minimum dominating set

(will be used for bounding cop number of random graphs)

Cop number of trees

The cop number of any tree is one.



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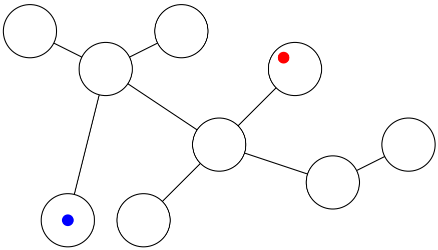

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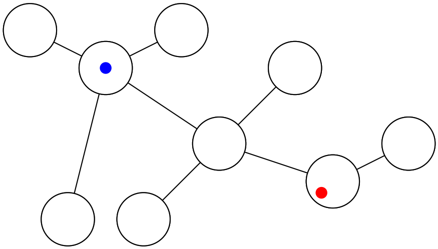

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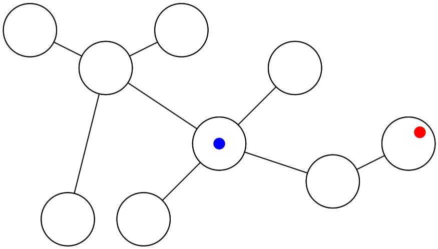

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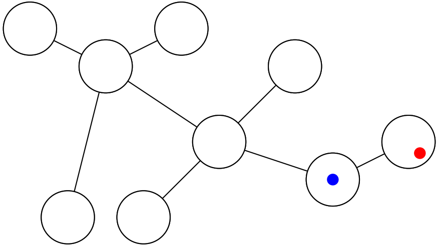
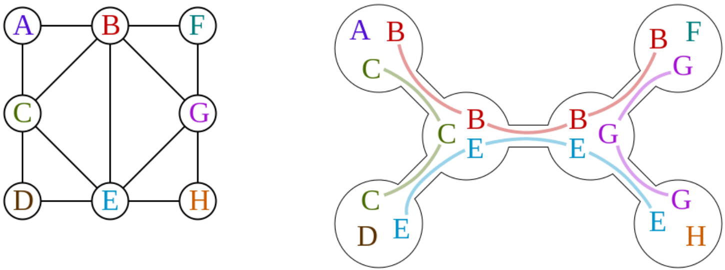
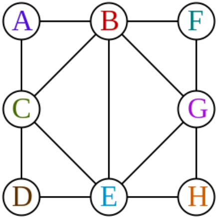


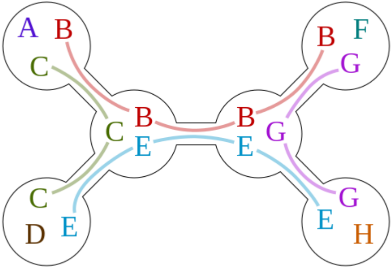
Tree decompositions



1. for every edge of graph there is a bag of tree containing both endpoints.
2. Each vertex of graph induces a connected subtree in the tree.

Tree decompositions: treewidth



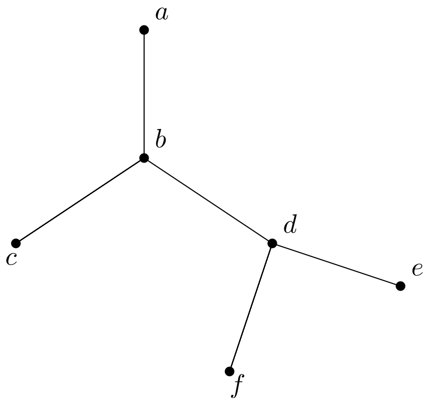


Width = maximum size of a bag $-1 = 2$

$\mathrm{tw}(G)$ = minimum width of a tree decomposition for G

Tree decomposition of a tree

Treewidth of a tree is 1



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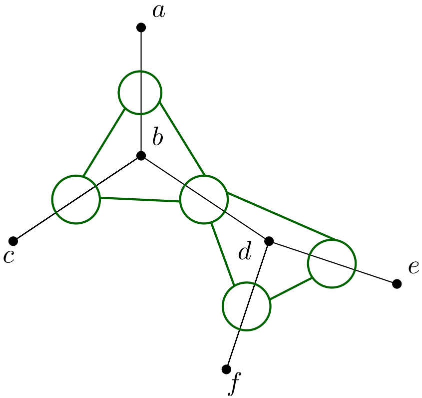

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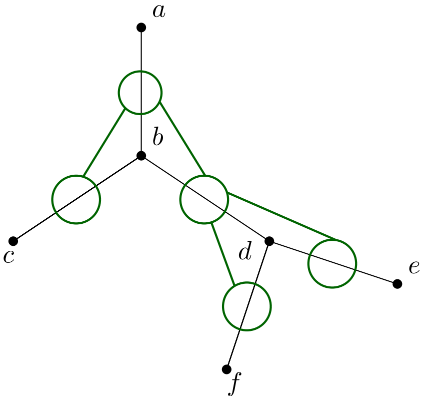

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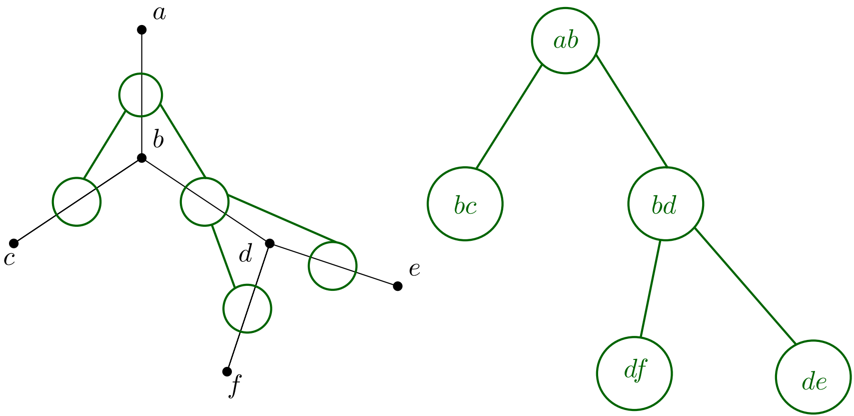
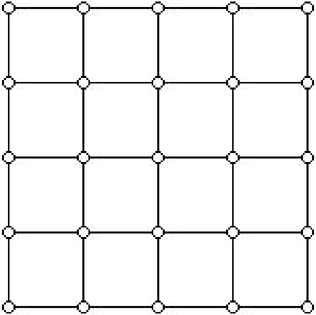


Examples of treewidth

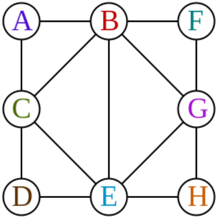
Example

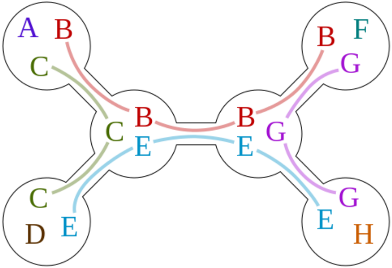
1. Treewidth of a complete graph is $n - 1$
2. Treewidth of a planar graph is $\leq L\sqrt{n}$
3. Treewidth of the $m \times m$ grid is m



The Relation Between Cop Number and Treewidth

For any G, $c(G) \leq \text{tw}(G) + 1$





Two easy upper bounds

For any graph G we have

$$c(G) \leq \min\{\gamma(G), \mathrm{tw}(G) + 1\}$$

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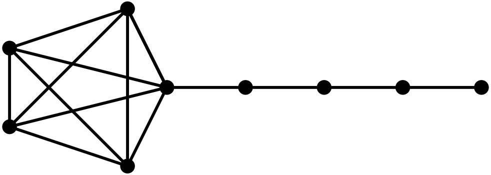

$$\mathrm{tw}(G) = \frac{n}{2} - 1, \gamma(G) \geq n/6$$
$$c(G) = 1$$

Our main result

Theorem (Alon, M'15)

For any G

$$\frac{\mathrm{tw}(G)+1}{\Delta(G)+1} \leq c(G) \leq \mathrm{tw}(G)+1$$

There exists a constant $\lambda > 0$ such that for a random graph $G = \mathcal{G}(n,p)$, with high probability,

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The two easy upper bounds are tight up to a constant factor, for two important classes of graphs.

Bounded-degree graphs

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Helicopter Cops and Robber Game

- ✓ A continuous-time game.
- ✓ At any moment, the robber is at a vertex.
- ✓ At any moment, each cop is either standing at a vertex, or in a helicopter.
- ✓ The cops want to land via a helicopter on the robber's vertex.
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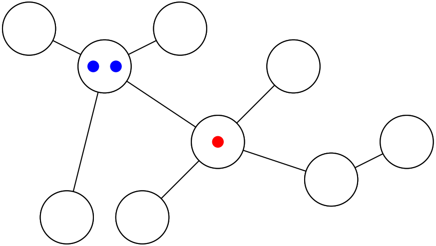
In a complete graph, n cops are needed.

Helicopter game on a tree



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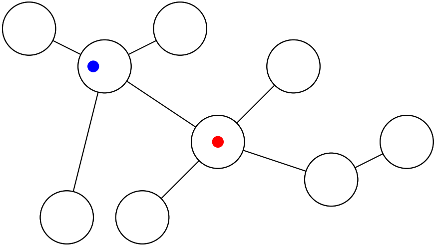

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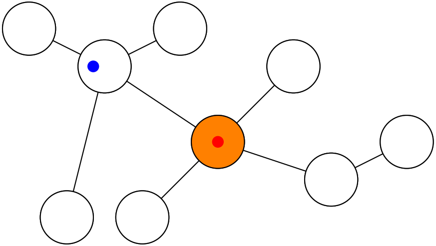

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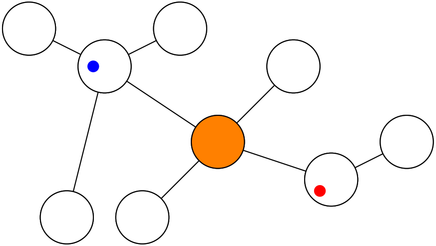

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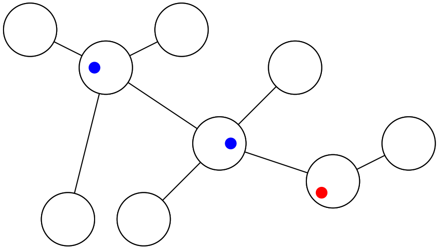

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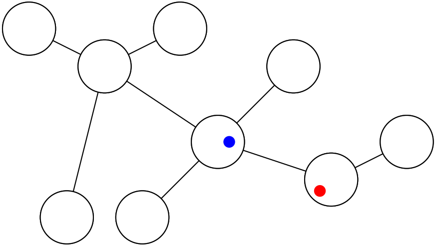

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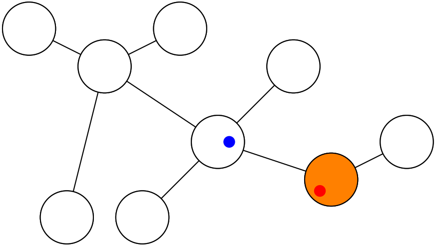

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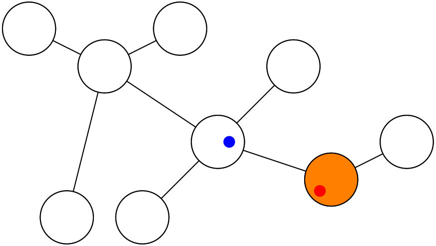

Helicopter cop number of a tree is 2

A Lower Bound for Cop Number

Theorem (Seymour and Thomas'93)

Exactly $\mathrm{tw}(G) + 1$ *cops are needed to capture the robber in the Helicopter Cops and Robber game.*

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number of required cops is at most multiplied by $\Delta + 1$.
Hence:

$$(\Delta + 1)c(G) \geq \mathrm{tw}(G) + 1$$

What we've proved so far

Proposition

For any graph G we have

$$\frac{\mathrm{tw}(G)+1}{\Delta(G)+1} \leq c(G) \leq \mathrm{tw}(G)+1$$

Complete graph: treewidth $=$ max. degree $= n-1$.

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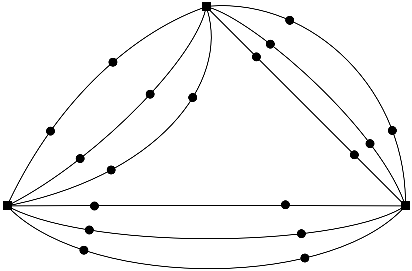

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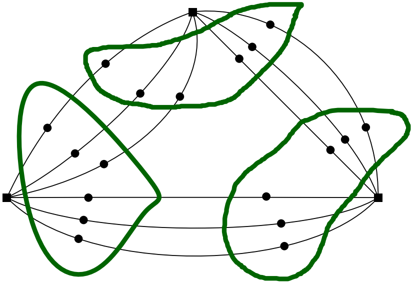

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Random graphs

The Random graph model

Definition

$\mathcal{G}(n,p)$ is a random graph on n vertices,
each edge appears in $\mathcal{G}(n,p)$ with probability p
(p can depend on n).
For a graph property $\mathcal{A}$, we say $\mathcal{G}(n,p)$ asymptotically almost surely (a.a.s.) satisfies $\mathcal{A}$, if

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$$\gamma(G) \geq c(G) \geq \text{number of isolated vertices} \geq ne^{-L}/2 \geq \gamma(G)e^{-L}/2$$

Next we analyze the case $\frac{1}{n} \ll p$, i.e. $pn \to \infty$

Main lemma for random graphs

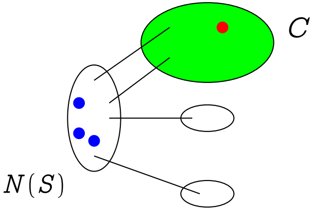
Lemma

Suppose $d := pn \to \infty$ *as* $n \to \infty$. *For any fixed* $\varepsilon > 0$ *we have*

$$(1-\varepsilon)\frac{\ln(d)}{d} \times n < c(G) \leq \gamma(G) < (1+\varepsilon)\frac{\ln(d)}{d} \times n$$

Escaping strategy for the robber

$N(S) :=$ (closed) neighbourhood of S



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S = cops' position

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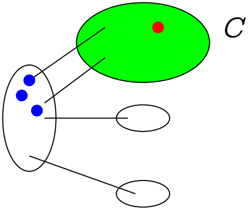

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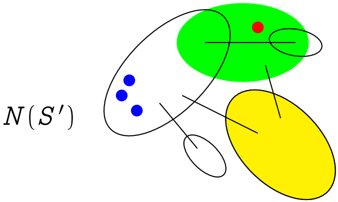

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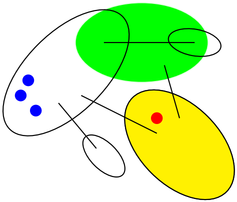

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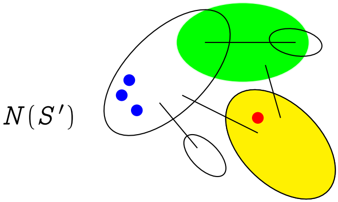

Invariant: Robber in largest component of $G - N(S')$
$S' =$ cops' position
Need 2 things:
(1) if S is small, largest component of $G - N(S)$ is big,
(2) and there is an edge between any two big subsets of vertices

Lower bound for cop number of random graphs

Principle: a.a.s. for all large sets X, $|V(G) \setminus N(X)|$ is very close to its expected value $= (1-p)^{|S|} \approx e^{-p|S|}$.

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Fact 1: a.a.s. any two subsets of size b are joined by an edge.
Bus problem: if you have some numbers summing to $\geq 3b$, one of the following is true:
one of them is at least b,
or you can partition them into two groups, each group summing to $\geq b$.

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Fact 2: a.a.s. any S with $|S| \leq (1-\varepsilon)n \ln d/d$ has $|V(G) \setminus N(S)| \geq 3b$.

Lower bound for cop number of random graphs

Principle: a.a.s. for all large sets X, $|V(G) \setminus N(X)|$ is very close to its expected value $= (1-p)^{|S|} \approx e^{-p|S|}$.

Set $b := en \ln d/d$.
Fact 1: a.a.s. any two subsets of size b are joined by an edge.
Bus problem: if you have some numbers summing to $\geq 3b$, one of the following is true:
one of them is at least b,
or you can partition them into two groups, each group summing to $\geq b$.
Hence, a.a.s. any set of vertices of size $\geq 3b$ has a connected component of size $\geq b$.

Fact 2: a.a.s. any S with $|S| \leq (1-\varepsilon)n \ln d/d$ has $|V(G) \setminus N(S)| \geq 3b$.
Therefore, $c(G) > (1-\varepsilon)n \ln d/d$

Upper bound for domination number of random graphs

Claim: for any graph G with minimum degree δ,

$$\gamma(G) \leq \frac{1 + \ln \delta}{\delta} \times n$$

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An adaptation of this proof gives a.a.s.

$$\gamma(\mathcal{G}(n, p)) < (1 + \varepsilon)\frac{\ln(d)}{d} \times n$$

What we've proved for random graphs

Lemma

Suppose $d := pn \to \infty$ *as* $n \to \infty$. *For any fixed* $\varepsilon > 0$ *we have*

$$(1-\varepsilon)\frac{\ln(d)}{d} \times n < c(G) \leq \gamma(G) < (1+\varepsilon)\frac{\ln(d)}{d} \times n$$

Combining with the analysis for the constant d case, we conclude the following.

Theorem (Alon, M'15)

There exists a constant $\lambda > 0$ *such that for a random graph* $G = \mathcal{G}(n,p)$, *asymptotically almost surely,*

$$\lambda\gamma(G) \leq c(G) \leq \gamma(G)$$

Conclusion

Summary of our results

Proposition (for all graphs)

$$\frac{\text{tw}(G)+1}{\Delta(G)+1} \leq c(G) \leq \min\{\text{tw}(G)+1, \gamma(G)\}$$

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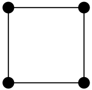
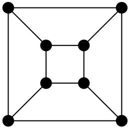
Question: find other graph classes for which these upper bounds are tight.

Hypercube graph

1-cube

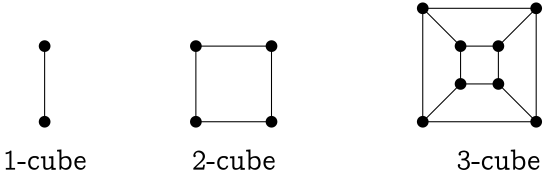


2-cube



3-cube

Hypercube graph



Proposition

For any graph G we have

$$\frac{\mathrm{tw}(G)+1}{\Delta(G)+1} \leq c(G) \leq \gamma(G)$$

The d-cube graph has

$$\lambda_1 \times \frac{2^d}{d\sqrt{d}} \leq c(G) \leq \lambda_2 \times \frac{2^d}{d}$$