On the Density of Nearly Regular Graphs with a Good Edge-Labelling

Abbas Mehrabian

amehrabi@uwaterloo.ca

University of Waterloo

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What is a *Good Edge-Labelling*?

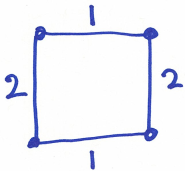
Graphs are simple and undirected and have n vertices.

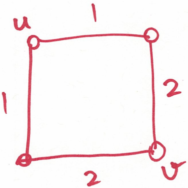
Definition (Bermond, Cosnard, and Pérennes 2009)

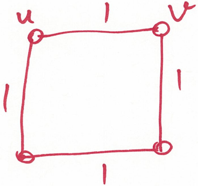
A good edge-labelling is a labelling of edges with integers such that for any ordered pair (u, v),
there is at most one increasing (non-decreasing) (u, v)-path.

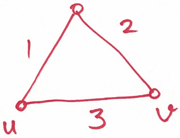
Defined in the context of Wavelength Division Multiplexing problems.

Example









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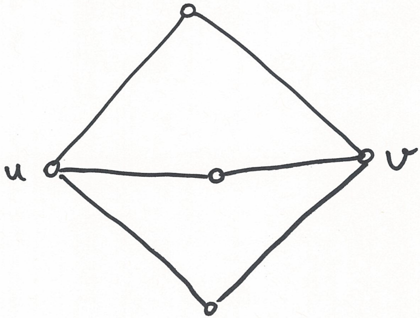

$K_{2,3}$

What is a *good* graph?

Definition

A graph is good if it admits a good edge-labelling; otherwise it is bad.

Example

C_4 is good, K_3 and $K_{2,3}$ are bad.

Density of Good Graphs

Question

What is the maximum number of edges of a good graph?

Araújo, Cohen, Giroire, and Havet (2009):

$$\Omega(n \log n) \leq \gamma(n) \leq O(n^{3/2}) \,.$$

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Today we will see that a good regular graph has $n^{1+o(1)}$ edges.

Girth of Bad Graphs

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Is having a small girth the obstacle for being good?

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Is having a small girth the obstacle for being good?

NO! We will see there exist bad graphs with arbitrarily large girth.

The Main Result

Theorem (M 2012+)

For any integer t, there exists $\epsilon(t)$ such that any d-regular n-vertex graph with $\epsilon(t)d^t > n$ is bad.

The Approach

Consider an arbitrary labelling of the graph,
and show there exist $> n^2$ increasing paths.

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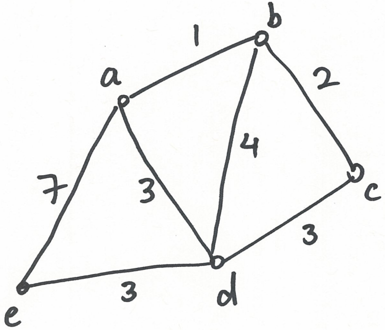
Definition

A nice k-walk is an increasing non-backtracking walk of length k.

Example



abcd

cba

adea

bcdc

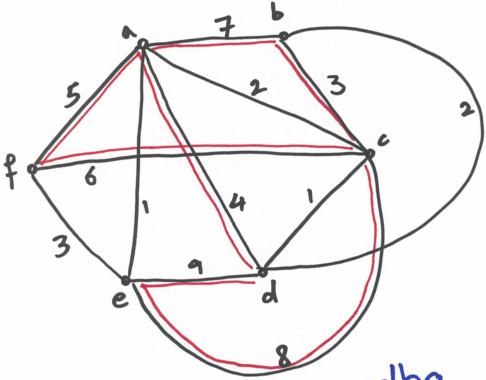
bd

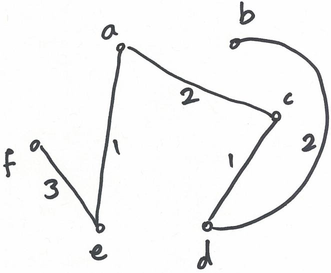
db

An Observation

If for some k there exist $> n^2$ nice k-walks, then the labelling is not good.

The Strategy





cdba
cdbc ← cdb

25 nice 3-walks ← 5 nice 2-walks

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Lemma

For any integer t, there exists $\epsilon(t) > 0$ such that any d-regular n-vertex graph has at least $\epsilon(t)nd^t$ nice t-walks.

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Can be extended to graphs with bounded Δ / $\overline{d}$.

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Let $\gamma_r(n)$ *be the maximum number of edges of a good regular graph. Then*

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Consider a sequence of regular graphs with at least $n^{1+\frac{1}{k}}$ edges, k fixed. Then their degree is going to infinity with n.
Consider a d-regular graph in this sequence with $d > 1/\epsilon(k+1)$.
Then for this graph

$$\epsilon(k+1)d^{k+1} > d^k = \left(\frac{2n^{1+\frac{1}{k}}}{n}\right)^k = 2^k n > n . \quad \square$$

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for any odd prime power d, there exists a d-regular graph with girth g with $< 2d^{\frac{3}{4}g}$ vertices.
Let $d > 2/\epsilon(g)$. Then

$$\epsilon(g)d^g > 2d^{g-1} > 2d^{\frac{3}{4}g} > n\,. \quad \square$$

A Result in The Other Direction

Theorem (M 2012+)

Any graph with max degree Δ and girth $\geq 2k$ such that

$$4ek^2(\Delta - 1)^{k-1} < k!$$

is good.

Corollary

Any graph with max degree Δ and girth $\geq 40\Delta$ is good.

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2. Labelling not good $\Rightarrow \exists$ increasing path of length exactly k (k-path).
3. For any k-path P, $\mathbf{Pr}[P \text{ increasing}] = \frac{2}{k!}$.
4. Any k-path intersects at most $2k^2(\Delta - 1)^{k-1}$ other k-paths. $\square$

Open Problems

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Open Problem 2 (Araújo, Cohen, Giroire, and Havet)

Does every planar graph of girth at least 5 have a good edge-labelling?

Thank you Slide

Thanks for your attention :-)