Randomized Rumour Spreading on Random k-trees

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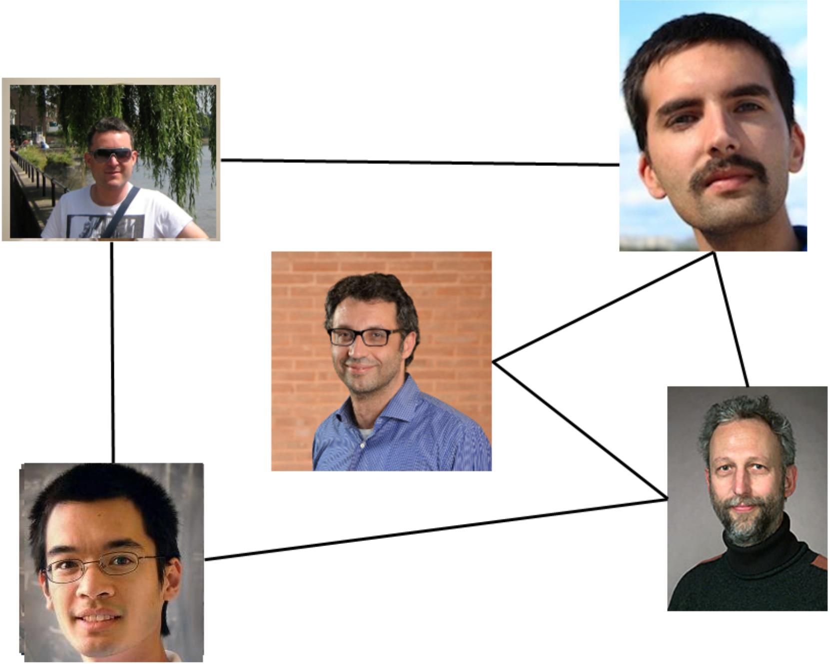
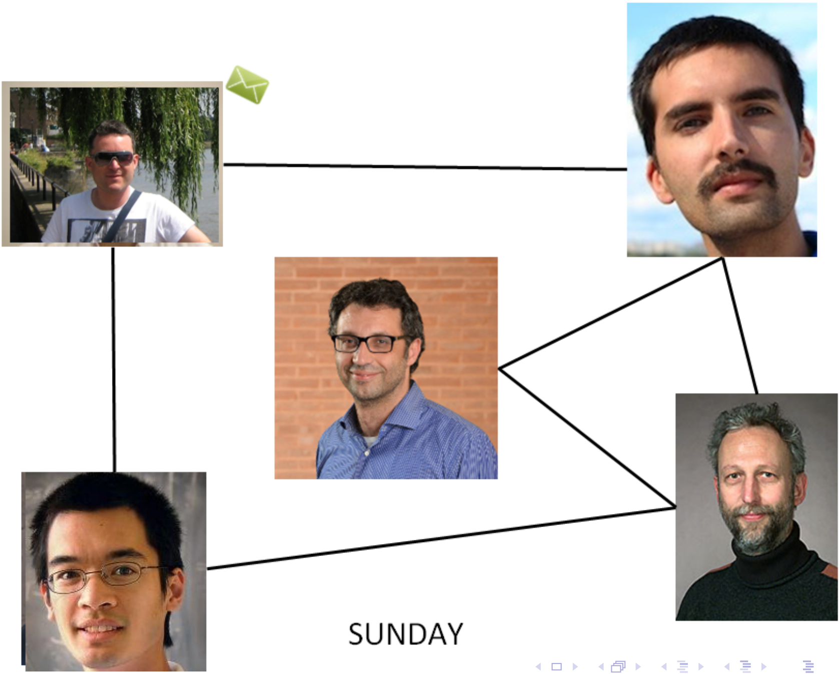
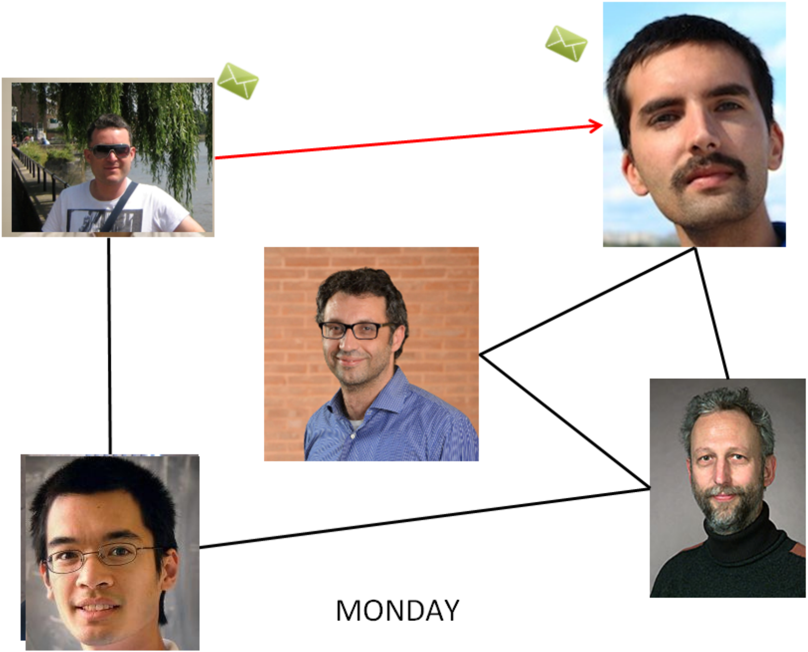
University of Melbourne

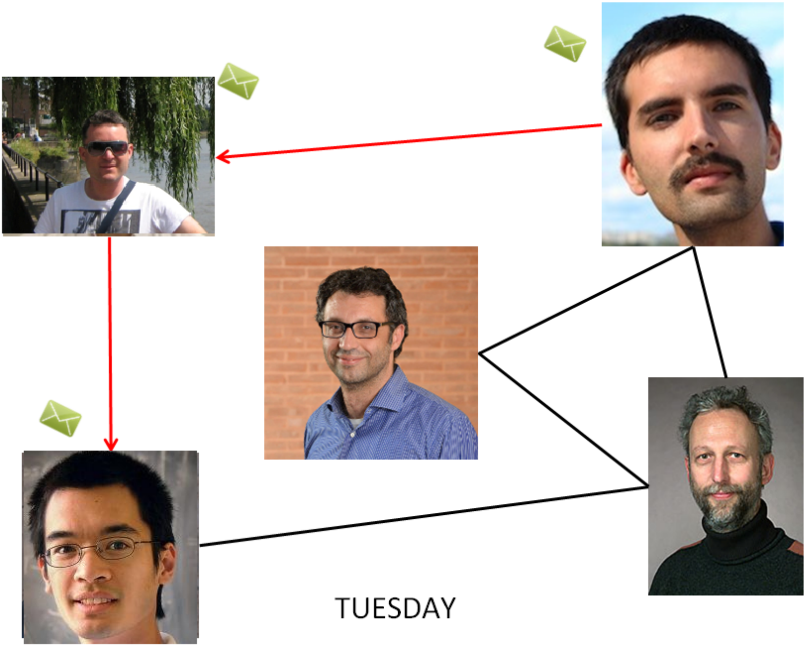




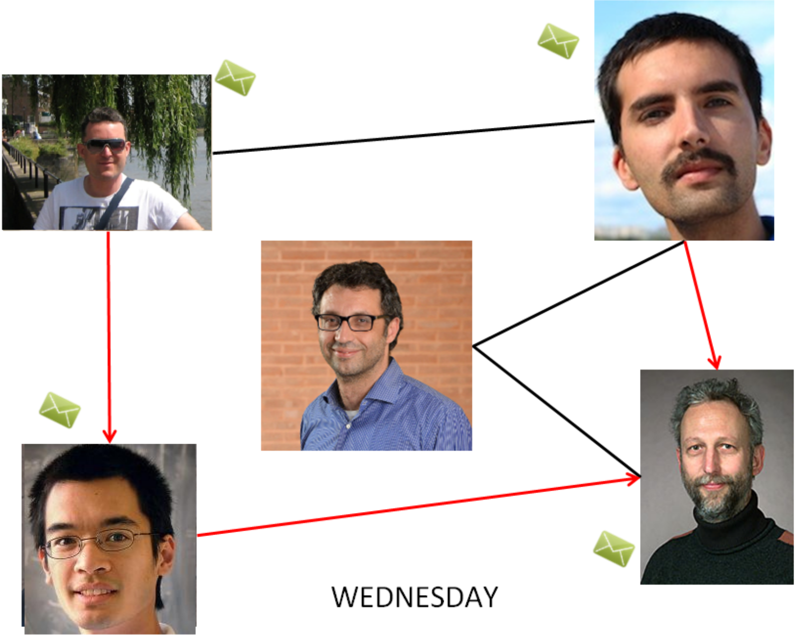
SUNDAY



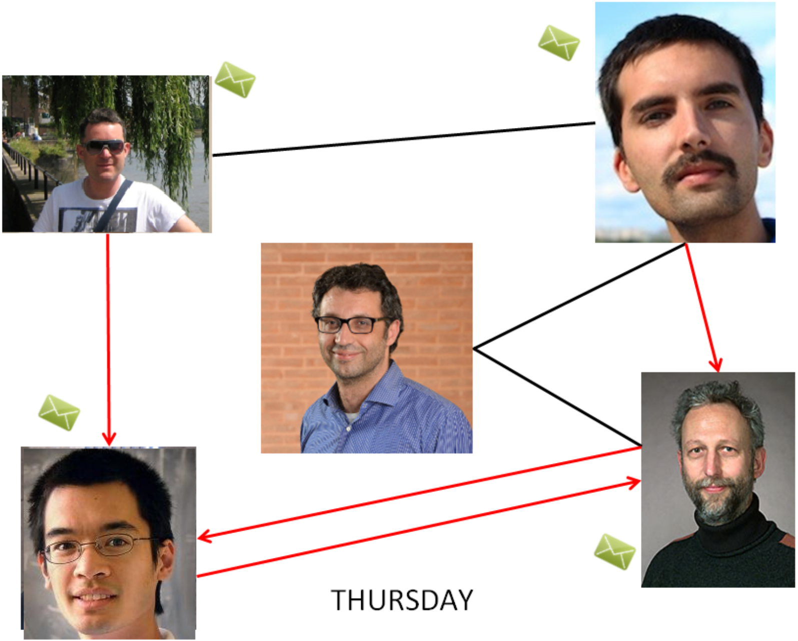
MONDAY

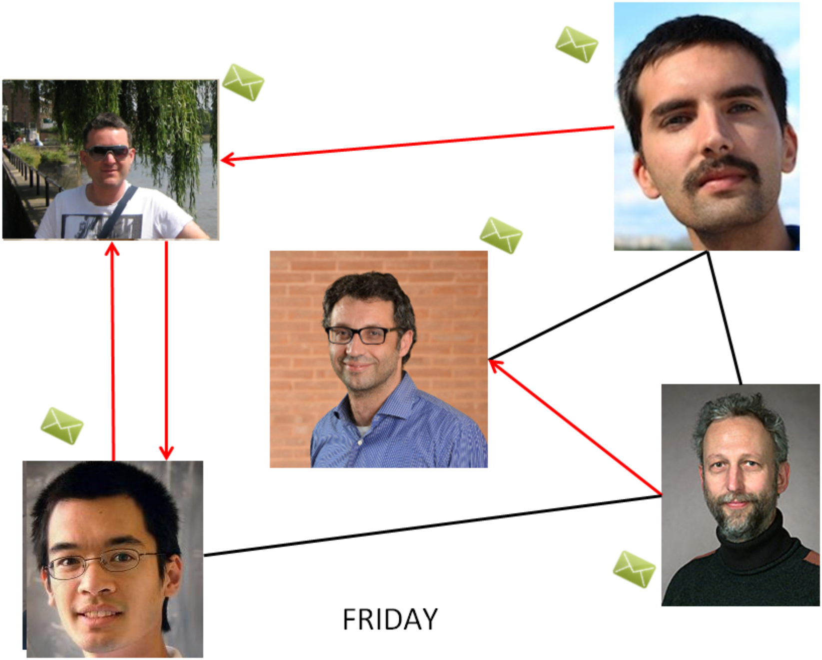


TUESDAY



WEDNESDAY


THURSDAY


FRIDAY

Protocol definition

Demers, Gealy, Greene, Hauser, Irish, Larson, Manning, Shenker, Sturgis, Swinehart, Terry, Woods'87

1. The ground is a simple connected graph.
2. At time 0, one vertex knows a rumour.
3. At each time-step $1, 2, \ldots,$ every informed vertex tells the rumour to a random neighbour.

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Remark 2. If a vertex receives the rumour at time t, it starts passing it from time $t + 1$.

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Remark 1. Informed vertex may call a neighbour in consecutive steps.
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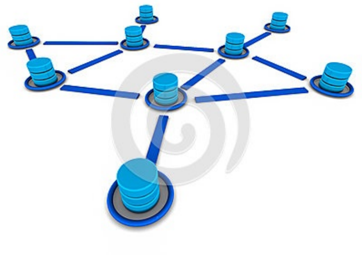
inform-time(v): the first time v learns the rumour.
Spread Time: the first time everyone knows the rumour.

Application: distributed computing



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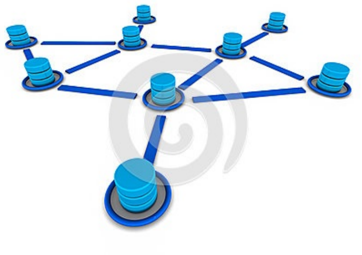
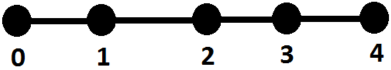


Rumour spreading advantages:

- ✓ Simplicity, locality, no memory
- ✓ Scalability, reasonable link loads
- ✓ Robustness

Example: a path



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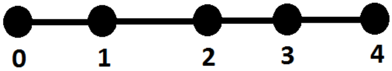
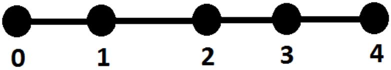


$$\mathrm{inform}-\mathrm{time}(0)=0$$

Example: a path



$$\text{inform} - \text{time}(0) = 0$$

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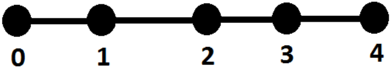
Example: a path



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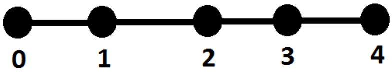

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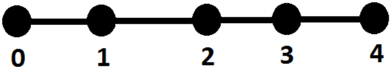

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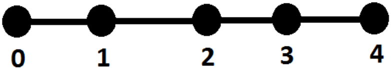

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$$\mathbb{E}[\text{Spread Time}] = 1 + 3 \times 2 = 7$$

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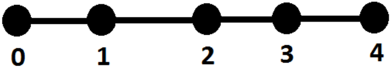

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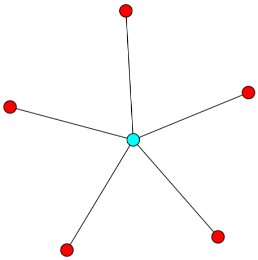
\mathbb{E}[\text{Spread Time}] &= 1 + 3 \times 2 = 7 \\
&= 2n - 3
\end{aligned}$$

Example: a star



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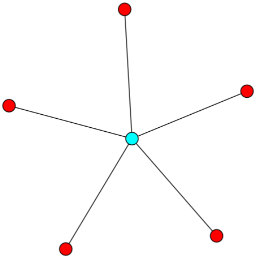

When $k + 1$ vertices are informed and $n - 1 - k$ uninformed, after $\frac{n-1}{n-1-k}$ more rounds a new vertex will be informed.

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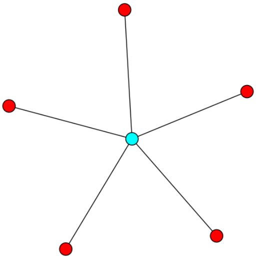
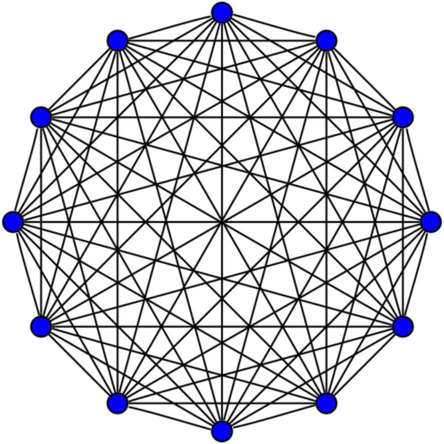


When $k+1$ vertices are informed and $n-1-k$ uninformed, after $\frac{n-1}{n-1-k}$ more rounds a new vertex will be informed.

$$\mathbb{E}[\text{Spread Time}] = \frac{n-1}{n-1} + \frac{n-1}{n-2} + \cdots + \frac{n-1}{2} + \frac{n-1}{1} \approx n \ln n$$

Example: a complete graph



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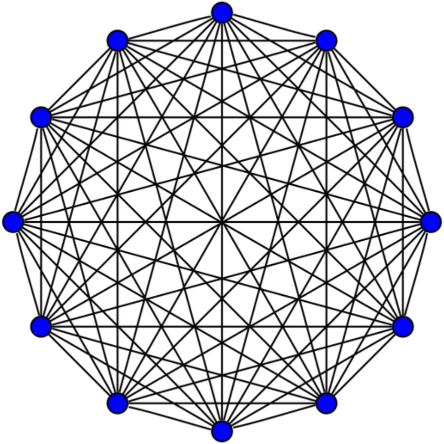

$$\mathbb{E}[\text{Spread Time}] \approx \log_2 n + \ln n \qquad [Pittel'87]$$

Other known results

With probability approaching 1, for any starting vertex,

1. $\max\{\text{diameter}(G), \log_2 n\} \leq \text{Spread Time} \leq (1 + o(1))n \ln n$
   [Elsässer and Sauerwald'06]

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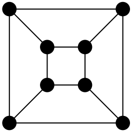
1. $\max\{\text{diameter}(G), \log_2 n\} \leq$ Spread Time $\leq (1 + o(1))n \ln n$ [Elsässer and Sauerwald'06]
2. Spread Time of $\mathcal{H}_d = \Theta(d)$ [Feige, Peleg, Raghavan, Upfal'90]



$\mathcal{H}_3$

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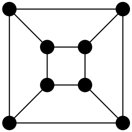

$\mathcal{H}_3$

3. If $pn \geq (1 + \varepsilon) \ln n$ then Spread Time of $G(n, p) = \Theta(\log n)$ [Feige et al.'90]

Improving the protocol

Uninformed vertices ask the informed ones...

The push-pull protocol

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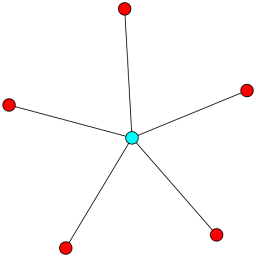
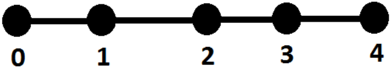


push protocol: $n \ln n$ rounds

push-pull protocol: 1 or 2 rounds

Example: a path



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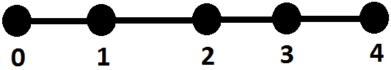

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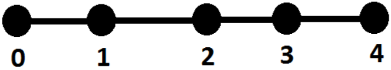

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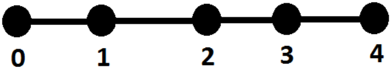

$$\mathrm{inform}-\mathrm{time}(0) = 0$$

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$$\begin{aligned}\mathrm{inform}-\mathrm{time}(2) &= 1+\min\{\mathrm{Geo}(1/2),\mathrm{Geo}(1/2)\}\\ &= 1+\mathrm{Geo}(3/4)\end{aligned}$$

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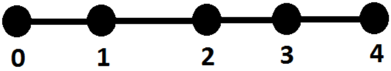

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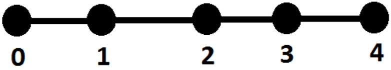

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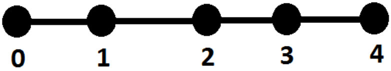
Example: a path



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$$\mathbb{E}[\text{Spread Time}] = 2 + 2 \times 4/3 = 14/3$$

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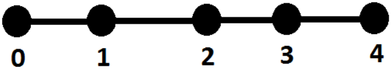

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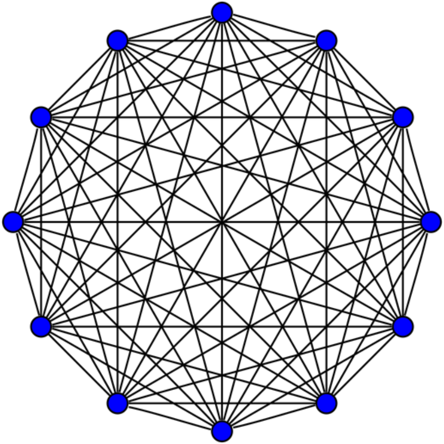
\mathbb{E}[\text{Spread Time}] &= 2 + 2 \times 4/3 = 14/3 \\
&= \frac{4}{3}n - 2 \qquad (\text{versus } 2n - 3 \text{ for push})
\end{aligned}
$$

Example: a complete graph



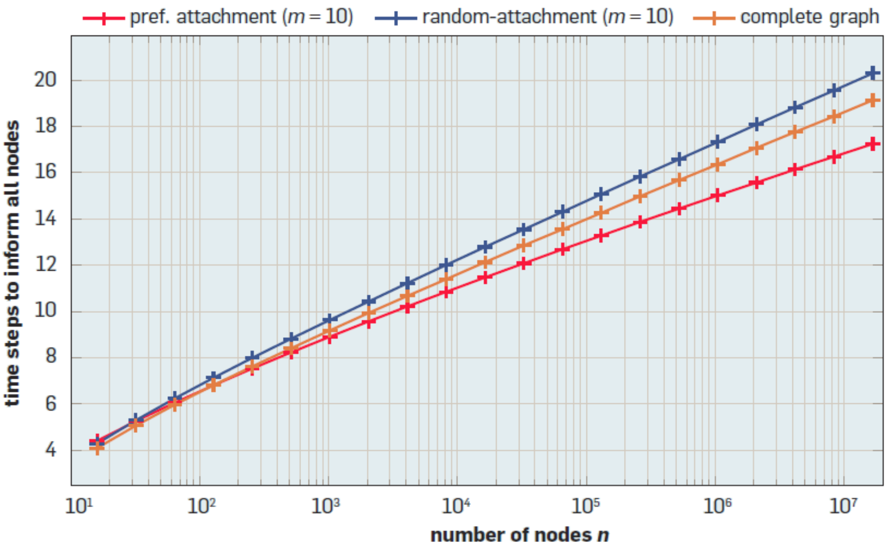
push: $\log_2 n + \ln n + o(\log n)$ [Pittel'87]

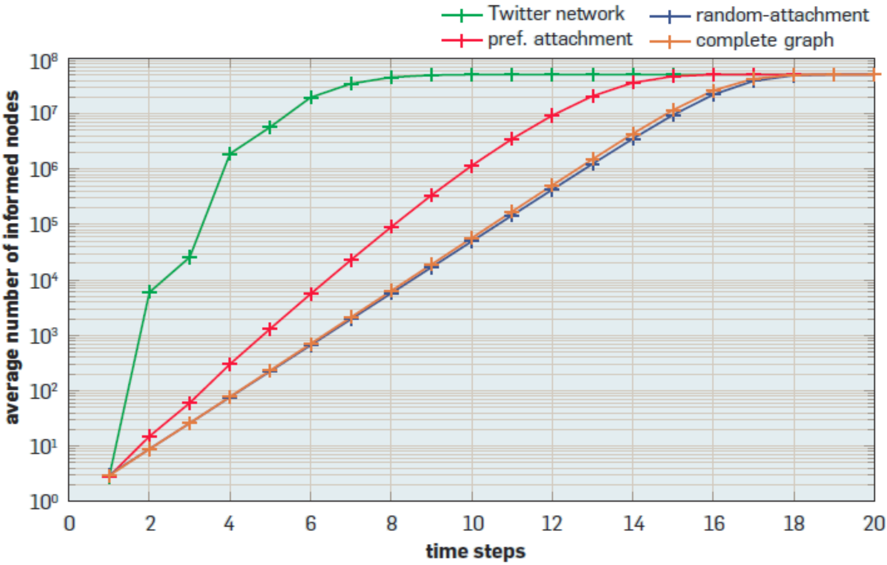
push-pull: $\log_3 n + o(\log n)$ [Karp, Schindelhauer, Shenker, Vöcking'00]

Other results on push-pull protocol

1. Barabasi-Albert preferential attachment graph has
   Spread Time $\Theta(\log n)$,
   PUSH alone has Spread Time $poly(n)$.
2. Random graphs with power-law expected degrees
   (a.k.a. the Chung-Lu model) with exponent $\in (2,3)$ has
   Spread Time $\Theta(\log n)$.
3. If Φ is Cheeger constant (conductance)
   and α is the vertex expansion (vertex isoperimetric number),
   Spread Time $\leq C \max\{\Phi^{-1} \log n, \alpha^{-1} \log^2 n\}$.

Let's see some simulation results...


pref. attachment ($m = 10$)
random-attachment ($m = 10$)
complete graph
time steps to inform all nodes
20
18
16
14
12
10
8
6
4
10^1
10^2
10^3
10^4
10^5
10^6
10^7
number of nodes n



$$n \approx 5 \times 10^7, \quad 64n \text{ edges}$$

Reference

Graphs in the previous slides were taken from:
Doerr, Fouz, Friedrich,
"Why rumors spread so quickly in social networks,"
Communications of the ACM, 2012

Some open problems

1. Design a (deterministic) approximation algorithm for finding the ‘average’ Spread Time of a given graph.

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Part II: Push-Pull on Random k-trees

joint work with Ali Pourmiri

Random k-trees

Example ($k = 3$)

Random k-trees

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Random k-trees

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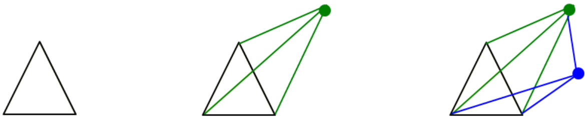
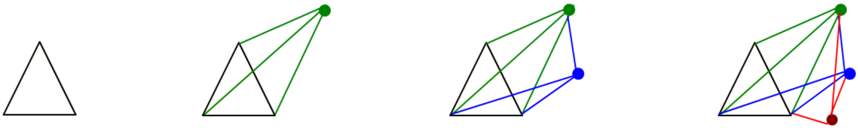
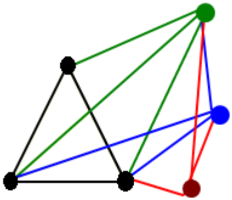


Random k-trees

Example ($k = 3$)



Some properties of random k-trees



- ✓ Logarithmic diameter
- ✓ Power law degree sequence:
  fraction of vertices with degree $d \approx d^{-2-\frac{1}{k-1}}$
- ✓ Constant tree-width k
- ✓ Clustering coefficient $\Omega(1)$
- ✓ Conductance and vertex expansion $o(1)$

Our results

Push-Pull protocol on random k-trees ($k > 1$ fixed):

Theorem (M, Pourmiri'14+)

If initially a random vertex knows the rumour,
a.a.s. after $\ln^{1+3/k} n$ *rounds,* $n - o(n)$ *vertices will know it.*

Our results

Push-Pull protocol on random k-trees ($k > 1$ fixed):

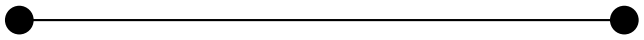
Theorem (M, Pourmiri'14+)

If initially a random vertex knows the rumour, a.a.s. after $\ln^{1+3/k} n$ *rounds,* $n - o(n)$ *vertices will know it.*

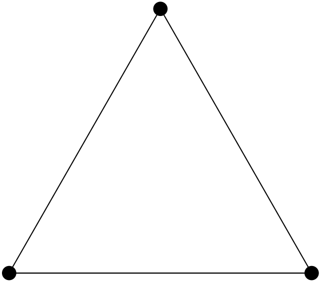
Theorem (M, Pourmiri'14+)

If initially a vertex knows the rumour, a.a.s. the average Spread Time is $> n^{1/3k}$

Self-similarity of random k-trees



Self-similarity of random k-trees



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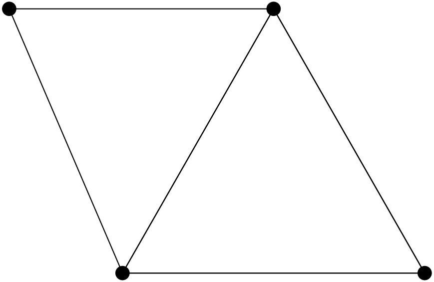

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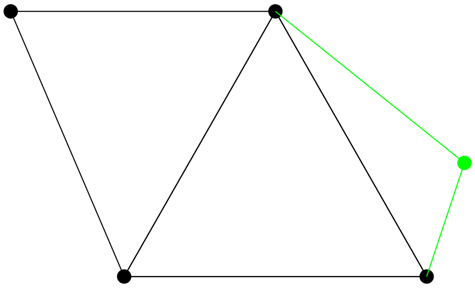

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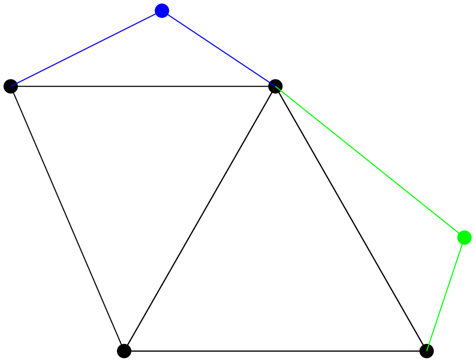

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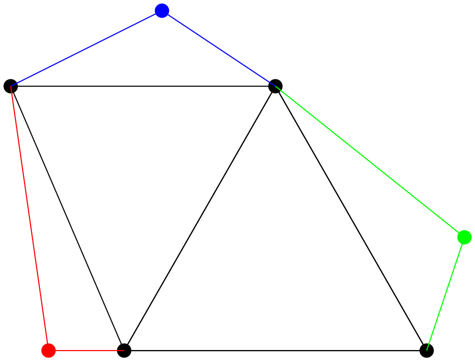

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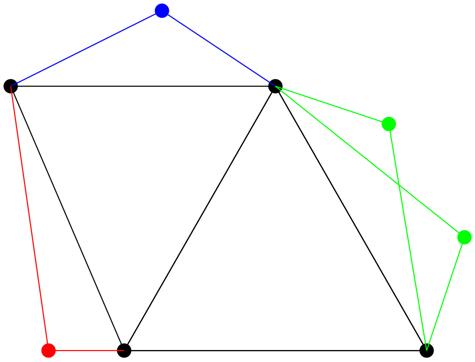

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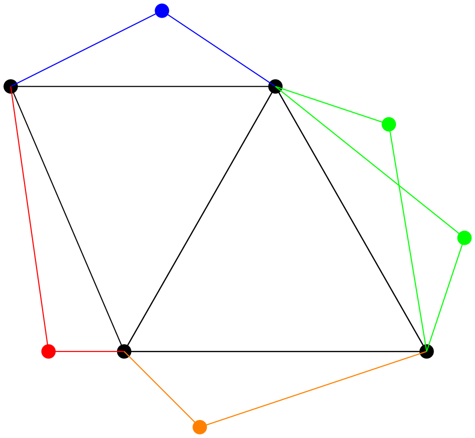

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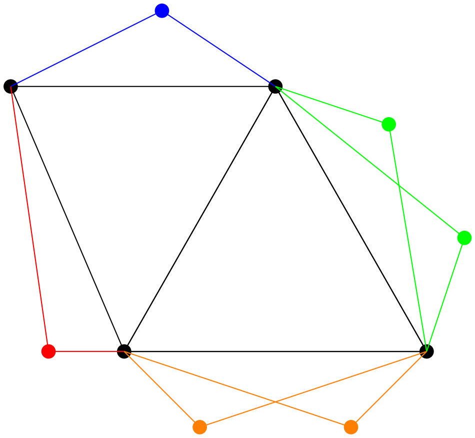

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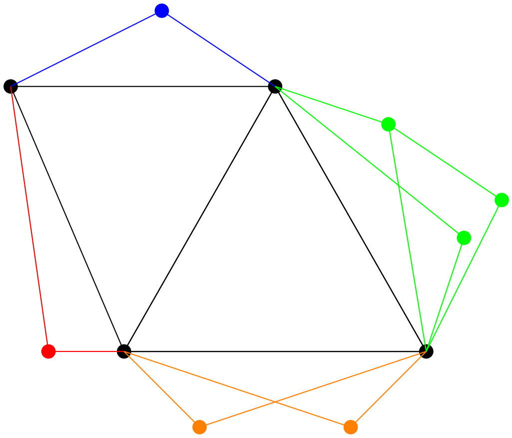

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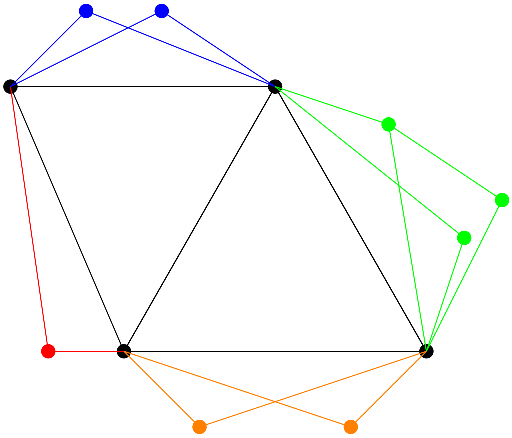

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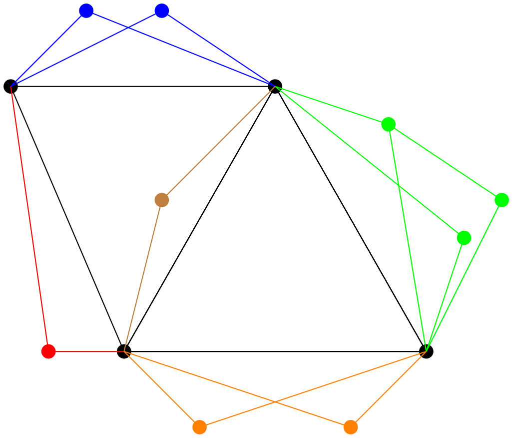

Proof of upper bound

Lemma

Suppose s knows the rumour at time 0, and $\exists(s, v)$-path $s = u_0, u_1, \ldots, u_{l-1}, u_l = v$ s.t. $\min\{\deg(u_i), \deg(u_{i+1})\} \leq d$. Then with prob. $\geq 1 - o(n^{-2})$, inform-time$(v) \leq 6d(l + \ln n)$.

Proof of upper bound

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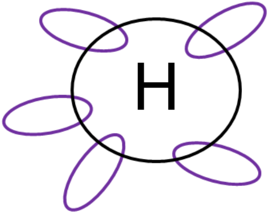
Proof.

$$
\begin{aligned}
&\mathrm{inform-time}(v) \\
&\leq \min\{\mathrm{Geo}(\frac{1}{d_0}), \mathrm{Geo}(\frac{1}{d_1})\} + \cdots + \min\{\mathrm{Geo}(\frac{1}{d_{l-1}}), \mathrm{Geo}(\frac{1}{d_l})\} \\
&\leq \text{sum of } l \text{ independent } \mathrm{Geo}(1/d) \text{ random variables}
\end{aligned}
$$

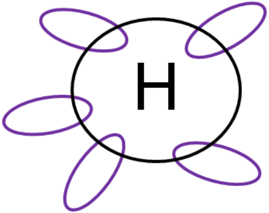
□

Proof of upper bound



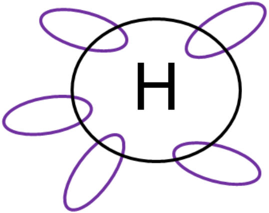
1. H = graph at round $\approx n \ln^{-2/k} n$

Proof of upper bound



1. H = graph at round $\approx n \ln^{-2/k} n$
2. Almost all vertices in small pieces have degrees $\leq d = \ln^{3/k} n$

Proof of upper bound



1. $H =$ graph at round $\approx n \ln^{-2/k} n$
2. Almost all vertices in small pieces have degrees $\leq d = \ln^{3/k} n$
3. An edge $uv \in E(H)$ is fast if $\deg(u) \leq d$ or $\deg(v) \leq d$ or u and v have a common neighbour with degree $\leq d$.

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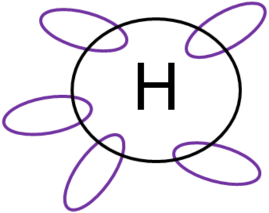


1. H = graph at round $\approx n \ln^{-2/k} n$
2. Almost all vertices in small pieces have degrees $\leq d = \ln^{3/k} n$
3. An edge $uv \in E(H)$ is fast if $\deg(u) \leq d$ or $\deg(v) \leq d$ or u and v have a common neighbour with degree $\leq d$.
4. $\exists$ an almost-spanning tree of H of height $O(\ln n)$ consisting of fast edges.

The upper bound

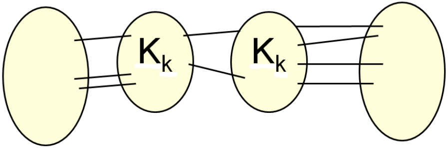
Push-Pull protocol on random k-trees ($k > 1$ fixed):

Theorem (M, Pourmiri'14+)

If initially a random vertex knows the rumour,
a.a.s. after $\ln^{1+3/k} n$ *rounds,* $n - o(n)$ *vertices will know it.*

Proof of lower bound

Definition (D-barrier)



Vertices in the two k-cliques have degrees $\geq D$.

Proof of lower bound

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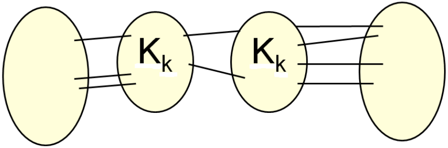
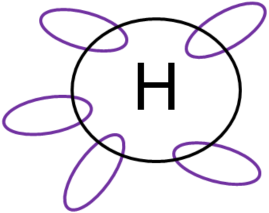


Vertices in the two k-cliques have degrees $\geq D$.

Lemma

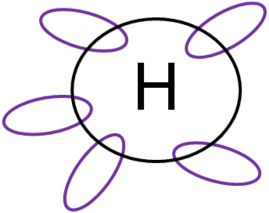
A random k-tree has a $\Omega\left(n^{1-1/k}\right)$-barrier with prob. $\geq \Omega\left(n^{-k}\right)$

Proof of lower bound



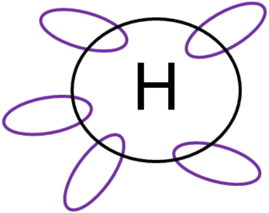
1. H = graph at round $m \approx n^{\frac{k}{k+1}}$

Proof of lower bound



1. H = graph at round $m \approx n^{\frac{k}{k+1}}$
2. Each small piece has a $(n/km)^{1-1/k}$-barrier with prob. $\Omega\left((n/km)^{-k}\right)$.

Proof of lower bound



1. H = graph at round $m \approx n^{\frac{k}{k+1}}$
2. Each small piece has a $(n/km)^{1-1/k}$-barrier with prob. $\Omega\left((n/km)^{-k}\right)$.
3. Since $km\left((n/km)^{-k}\right) \to \infty$ and by independence of pieces, with prob. $1 - o(1)$ there exists a $(n/km)^{1-1/k}$-barrier.

The lower bound

Push-Pull protocol on random k-trees ($k > 1$ fixed):

Theorem (M, Pourmiri'14+)

If initially a vertex knows the rumour,
a.a.s. the average Spread Time is $> n^{1/3k}$

Some open problems

1. Design a (deterministic) approximation algorithm for finding the 'average' Spread Time of a given graph.
2. These questions may be asked about the 'asynchronous' model.