Quantum query complexity of determining whether a graph is connected

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The connectivity problem

Design an algorithm that determines if a given graph is connected.

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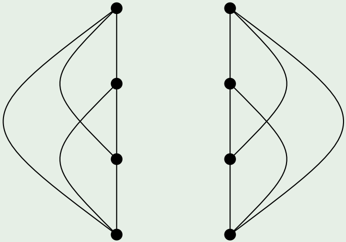
Example



NO

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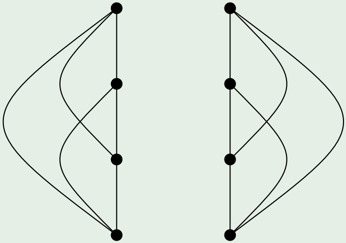
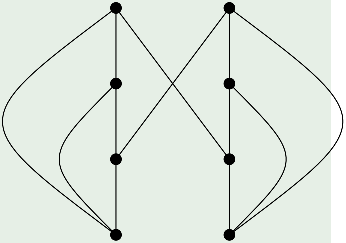
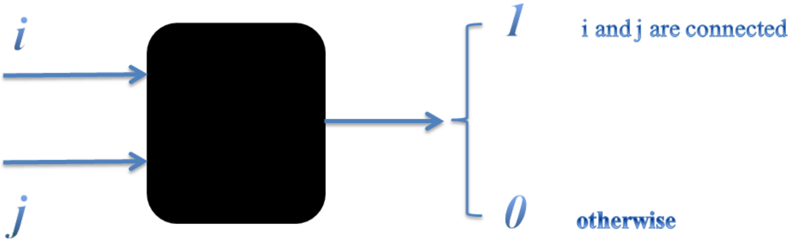


NO



YES

Assume the vertices are numbered from 1 to n.



Access to the input graph

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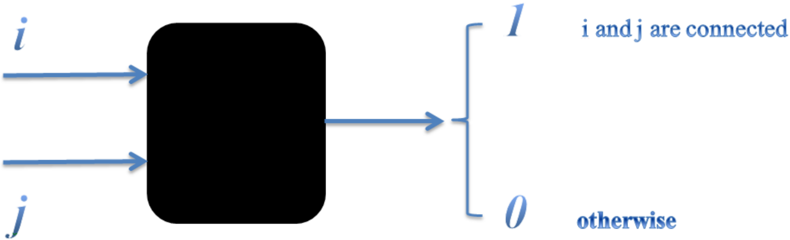

Each question asked from the black-box is called a query.

Theorem

There is a quantum algorithm with query complexity $O(n\sqrt{n})$ for the connectivity problem with error probability $< 1/3$.

Indeed, this is optimal up to constant factors.

Main results

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Theorem

Every classical algorithm for the connectivity problem that has error probability $< 1/3$ has query complexity $\Omega(n^2)$.

The Quantum Algorithm

Generalization of Grover's search algorithm

Theorem (Grover's search algorithm)

There is a quantum algorithm that given a domain of size N and black-box access to some function $F : \{1, \ldots, N\} \to \{0, 1\}$, for which it is guaranteed that there exists a solution y with $F(y) = 1$, finds a solution evaluating F at $O(\sqrt{N})$ points.

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There is a quantum algorithm that given a domain of size N *and black-box access to some function* $F : \{1, \dots, N\} \to \{0, 1\}$*, for which it is guaranteed that there exists a solution* y *with* $F(y) = 1$*, finds a solution evaluating* F *at* $O(\sqrt{N})$ *points.*

Theorem (Generalization of Grover's search algorithm)

There is a quantum algorithm that if there are s *solutions,*

- ✓ *If* $s > 0$*, outputs a random solution asking an expected number of* $O\left(\sqrt{N/s}\right)$ *queries.*
- ✓ *If* $s = 0$*, the algorithm does not halt.*

Note: the algorithm need not know s*.*

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- ✓ *If $s = 0$, the algorithm does not halt.*

Note: the algorithm need not know s.

This algorithm will be called the search algorithm.

Dealing with the never-stopping issue

We present an algorithm that given black-box access to G,

- ✓ if G is connected, outputs YES after asking $O(n\sqrt{n})$ queries on average; and
- ✓ does not halt if G is not connected.

Dealing with the never-stopping issue

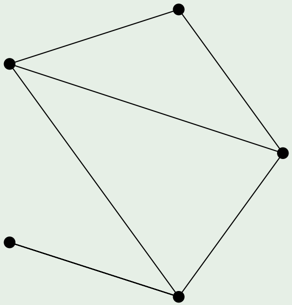
We present an algorithm that given black-box access to G,

- ✓ if G is connected, outputs YES after asking $O(n\sqrt{n})$ queries on average; and
- ✓ does not halt if G is not connected.

It is not hard to turn this into an algorithm with worst-case query complexity $O(n\sqrt{n})$ and error probability $< 1/3$.

The algorithm

Example



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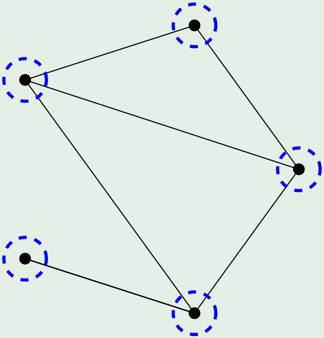

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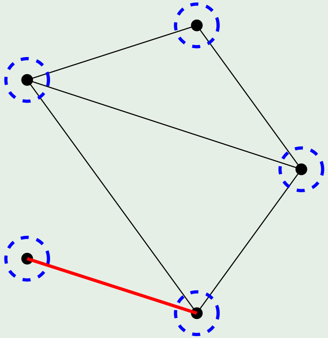

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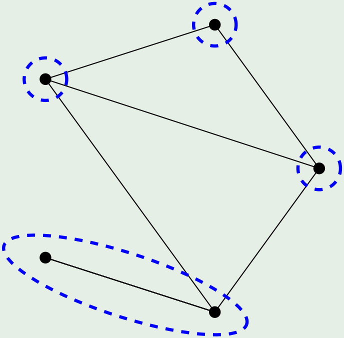

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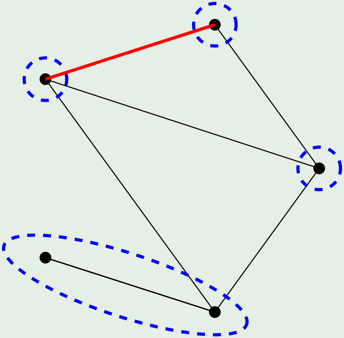

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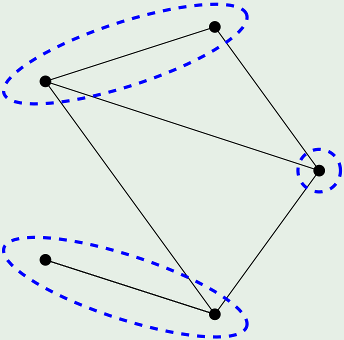

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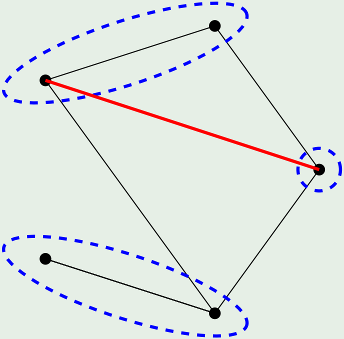

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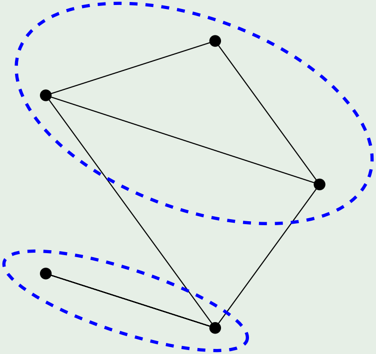

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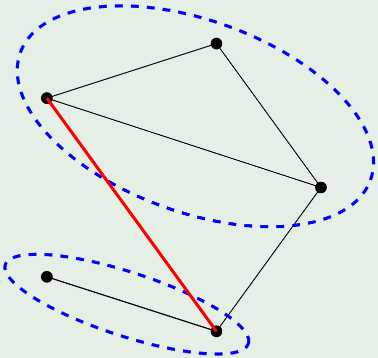

The algorithm

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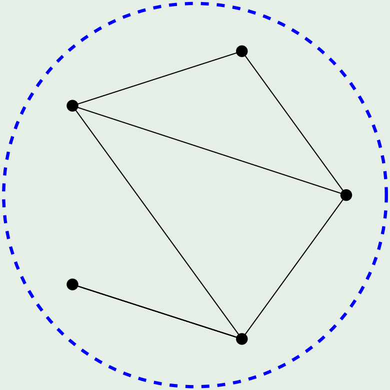
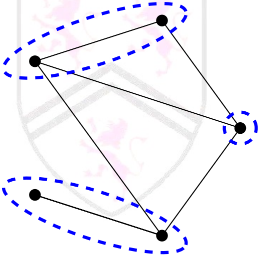


Analysis: connected case



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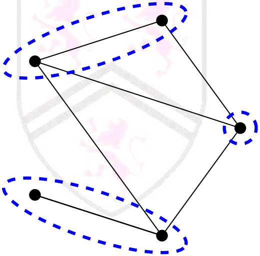
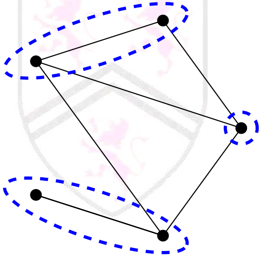


When there are k pieces, there are $\geq k-1$ desirable edges, and domain size is $\binom{n}{2}$. Hence average query complexity of the search algorithm is $O\left(\sqrt{\binom{n}{2} \Big/ (k-1)}\right) = O\left(\sqrt{n^2/k}\right)$.



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Average query complexity of the whole algorithm is

$$\sum_{k=2}^{n} O\left(\sqrt{n^2/k}\right) \leq O\left(n\sqrt{n}\right)$$

Quantum algorithm: the theorem

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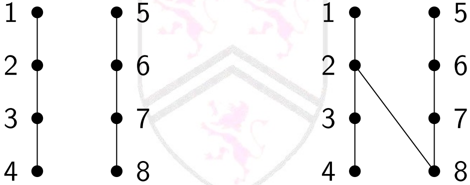
Classical lower bound



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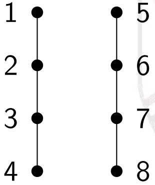
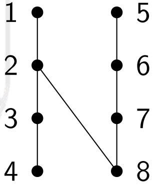




$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Classical lower bound: the theorem

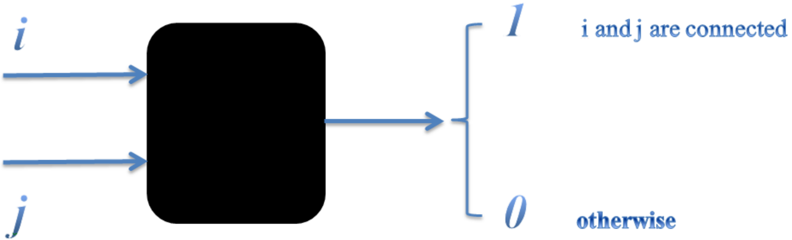
Theorem

Every classical algorithm for the connectivity problem that has error probability $< 1/3$ *has query complexity* $\Omega(n^2)$*.*

Another model

Recall: access to the input graph

Vertices are numbered from 1 to n.



This is called the matrix model.

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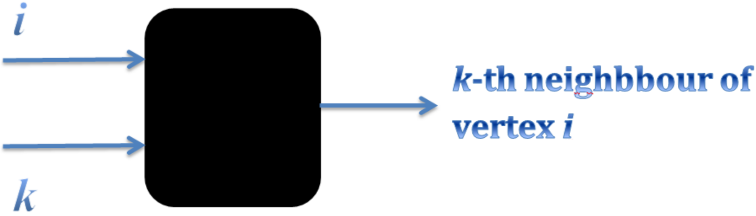
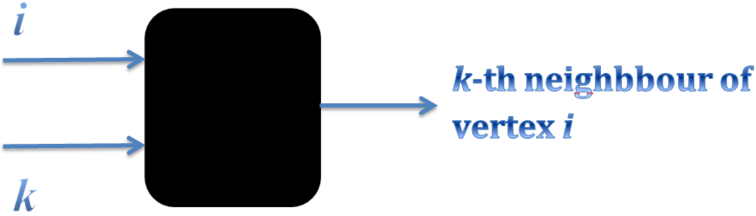


A different model

Vertices are numbered from 1 to n.



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Main results

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In the array model, there is a quantum algorithm with query complexity $O(n)$ for the connectivity problem with error probability $< 1/3$.

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Outline of the previous algorithm

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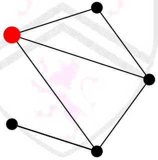
1. Partition the vertex set into connected pieces.
2. Merge the pieces one by one.

Some definitions

The degree of a vertex is its number of neighbours:



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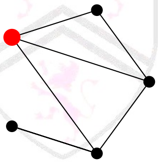
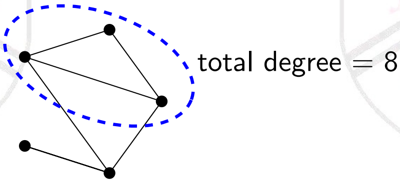
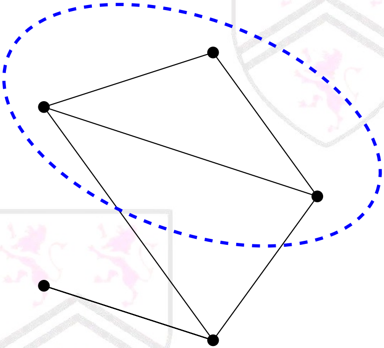


The total degree of a set of vertices S, written $t(S)$, is the sum of degrees of its vertices:



The key observation



total degree $= 8$

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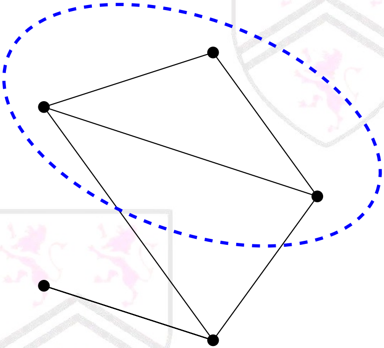

total degree = 8

In general, we need $O\left(\sqrt{t(P)}\right)$ queries to find an edge going out of P.

The algorithm

In the first phase, asking $O(n)$ classical queries we partition the vertices into connected pieces $P_1, P_2, \ldots, P_k$ such that $t(P_i) < |P_i|^2 \quad \forall i$.

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In the second phase, we merge the pieces iteratively: in every iteration,

1. choose a piece P with minimum total degree.
2. Use the search algorithm to find an edge going out of P.
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It turns out that if G is connected, then the expected query complexity of the second phase is

$$O\left(\sqrt{t(P_1)} + \sqrt{t(P_2)} + \cdots + \sqrt{t(P_k)}\right) \leq O(n)$$

The proved theorem

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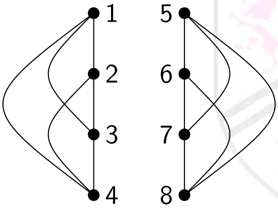
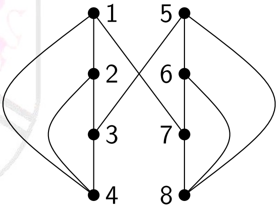
Classical lower bound





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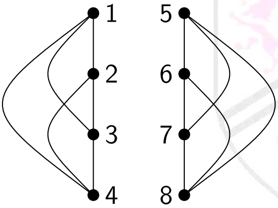

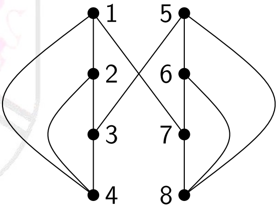

1	2	4	3
2	1	3	4
3	4	2	1
4	3	1	2
5	7	8	6
6	5	7	8
7	8	5	6
8	7	6	5

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6	5	7	8
7	8	1	6
8	7	6	5

Wrap-up

model	matrix	array
classical	$\Theta(n^2)$	$\Theta(n^2)$
quantum	$\Theta(n\sqrt{n})$	$\Theta(n)$

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Thank you