Proving logarithmic upper bounds for diameters of random graphs

Abbas Mehrabian

Pacific Institute for the Mathematical Sciences

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University of Saskatchewan

Motivation

Many random graphs have diameters at most $O(\log n)$ ...

Generator	Diameter or Avg path len.	Community		Clustering coefficient	Remarks
		Bip. core vs size	$C(k)$ vs k		
Erdös–Rényi [1960]	$O(\log N)$		Indep.	Low, $CC \propto N^{-1}$	
PLRG [Aiello et al. 2000], PLOD [Palmer and Steffan 2000]	$O(\log N)$	Indep.		$CC \to 0$ for large N	
Exponential cutoff [Newman et al. 2001]	$O(\log N)$			$CC \to 0$ for large N	
BA [Barabási and Albert 1999]	$O(\log N)$ or $O(\frac{\log N}{\log \log N})$			$CC \propto N^{-0.75}$	
Initial attractiveness [Dorogovtsev and Mendes 2003]					
AB [Albert and Barabási 2000]					
Edge copying [Kleinberg et al. 1999], [Kumar et al. 1999]		Power-law			
GLP [Bu and Towsley 2002]				Higher than AB, BA, PLRG	Internet only
Accelerated growth [Dorogovtsev et al. 2001], [Barabási et al. 2002]				Non-monotonic with N	
Fitness model [Bianconi and Barabási 2001]					
Aiello et al. [2001]					
Pandurangan et al. [2002]					
Inet [Winick and Jamin 2002]					Specific to the AS graph
Forest Fire [Leskovec et al. 2005]	"shrinks" as N grows				
Pennock et al. [2002]					
Small-world [Watts and Strogatz 1998]	$O(N)$ for small N, $O(\ln N)$ for large N, depends on p			$CC(p) \propto (1-p)^3$, Indep of N	N = num nodes p = rewiring prob
Waxman [1988]					
BRITE [Medina et al. 2000]	Low (like in BA)			like in BA	BA + Waxman with additions
Yook et al. [2002]					
Fabrikant et al. [2002]					Tree, density 1
R-MAT [Chakrabarti et al. 2004]	Low (empirically)				

Chakrabarti & Faloutsos, ACM Computing Surveys 2006.

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1. Why is it important?
2. What is the reason?
3. How to prove such results?

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Expansion? Works in many cases!

But, many random trees also have logarithmic diameter!

Our contribution

A technique for proving certain random graphs have diameter at most $O(\log n)$.

Note: no lower bounds today!

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The idea is to couple with a random recursive tree such that graph's diameter $\leq$ tree's diameter.

Definition (random recursive tree)

Fix k. Initially we have a single node;
in every round a uniformly random node gives birth to k new children.

Pittel'94 proved a.a.s. the height is asymptotic to $e \log n$.
Height: maximum distance between a vertex and the root

Example 1: Preferential attachment graphs

Model definition

1. Fix m and start with a connected graph.
2. In each step a new vertex and m new edges are born.
3. One endpoint of each new edge is the new vertex,
   the other endpoint is sampled according to the degrees.

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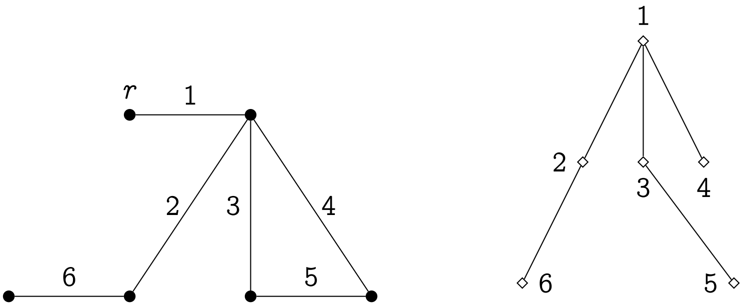
The diameter is a.a.s. $O(\log n)$ if $m = 1$ [Pittel'94] and $O(\log n / \log\log n)$ if $m > 1$ [Bollobás and Riordan'04].
We prove a.a.s. the diameter $\leq 4e \log n$.

Proof sketch

1. Choose a vertex of the initial graph as the root.
2. Couple with a tree whose nodes correspond to the edges of the graph, such that the depth of each node $\geq$ the distance between the corresponding edge and the root.



3. Creation of a new vertex (say $m = 3$)

Key observation: Sampling according to the degrees is equivalent to choosing a random endpoint of a uniformly random edge!

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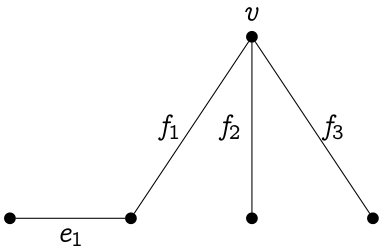

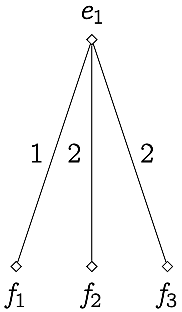

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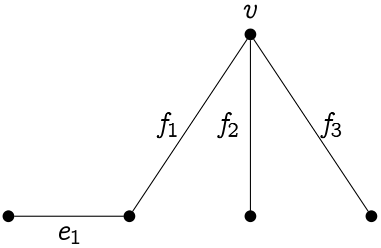

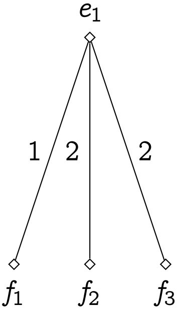

Thus, the diameter is a.a.s. at most $4e \log n$.

Slight generalization

What if the attachment probabilities are linear functions of the degrees?

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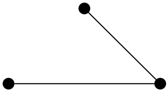
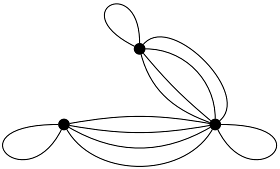
Say prob. of attaching to $v = 2\deg(v) + 1$





Example 2: The Cooper-Frieze model

The Cooper-Frieze model

- ✓ In an step, either a new vertex is born and edges are added from it to the existing graph,
  or edges are added between the existing vertices.
- ✓ The number of added edges is a bounded random variable.
- ✓ One endpoint of each added edge is either the new vertex,
  or a uniformly random vertex,
  or sampled according to degrees.
- ✓ The other endpoint is either a uniformly random vertex or sampled according to degrees.

[Cooper and Frieze'01]

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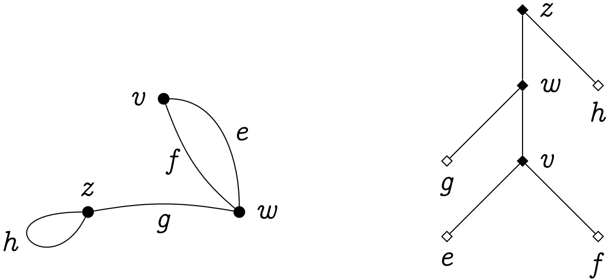
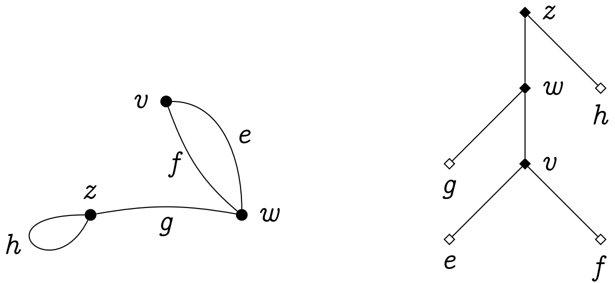
[Cooper and Frieze'01]

We prove a.a.s. the diameter is $O(\log n)$. (NEW RESULT!)

Proof sketch



Proof sketch



In every step, either a uniformly random white node gives birth, or a uniformly random black node gives birth.

The height is still logarithmic.

New results

Theorem (M'14+)

The following random graph models have diameter $O(\log n)$ a.a.s.

- ✓ *The (edge) copying model [Kumar,Raghavan,Rajagopalan,Sivakumar,Tomkins,Upfal'00]*
- ✓ *Aiello-Chung-Lu models* *[Aiello,Chung,Lu'01]*
- ✓ *The Cooper-Frieze model* *[Cooper,Frieze'01]*
- ✓ *The generalized linear preference model* *[Bu,Towsley'02]*
- ✓ *The PageRank-based selection model [Pandurangan,Raghavan,Upfal'02]*
- ✓ *Directed scale-free graphs* *[Bollobás,Borgs,Chayes,Riordan'03]*
- ✓ *The forest fire model* *[Leskovec,Kleinberg,Faloutsos'05]*

The PARID model of Deijfen, van den Esker, van der Hofstad and Hooghiemstra has diameter $O(\log^2 n)$ a.a.s. if the initial degrees' distribution has an exponential decay.