This is a summary of the talk I gave on May 10th of 2013 in Monash University for the “graph limits” reading group. The aim is to cover Sections 3.1–3.4 of the survey paper

Counting Graph Homomorphisms by Borgs, Chayes, Lovasz, Sos, and Vesztergombi (2006).

Abbas Mehrabian 12 May 2013

Sections 3.1 and 3.2 ... 3.3 and 3.4 (briefly 3.5) of Survey

Graphs are unweighted unless specified otherwise.

Definitions:

k-labelled graphs, (gluing) product, graph parameter, connection matrix

rank of infinite matrices, $rk(f,k)$, positive semi-definite (PSD)

Reflection positive/Positive semi-definite (PSD) parameter, Finite rank parameter

Lemma. Let $k \geq 0$ and f be a graph parameter. Assume there exists $g_1, g_2, \ldots, g_t, h_1, h_2, \ldots, h_t$ so that for every pair G, H of k-labelled graphs,

$$f(GH) = \sum_{i=1}^{t} g_i(G) h_i(H) \, .$$

Then $rk(f,k) \leq t$. If $g_i = h_i$ for $i = 1, 2, \ldots, t$ then $M(f,k)$ is PSD.

Proof. Write

$$M(f,K) = A_1 + \cdots + A_t$$

where

$$[A_i]_{G,H} = g_i(G) h_i(H)$$

Then draw A_i and show it has rank 1, so there is v_i that spans the row vector of A_i. Hence $\{v_1, \ldots, v_t\}$ span the row vector of $M(f,k)$.

Proof of PSD-ness is easy: just assign the vector $v_G = [g_1(G), \ldots, g_t(G)]$ to each graph G. $\square$

parameter	$rk(f,k)$	PSD
# edge	2	
# simple-edge	$2+\binom{k}{2}$	
# simple subgraphs	$2^{\binom{k}{2}}$	
# perfect matchings	2^k	NO
# independent sets	$finite$	YES
# hamiltonian cycles	$\leq 2^{k-1}(k-1)!$	YES
# q-colourings	q^k	YES
chr(.,x)	finite (2006)	
independence number	finite (2008)	
# spanning trees	finite	
max clique	∞	
chromatic number	∞	
# eulerian orientations	∞	YES

e.g. number of edges

$$e(GH) = e(G) \times 1 + 1 \times e(H)$$

e.g. number of edges of the simple version of G:

$$\begin{aligned} e'(GH) &= e'(G) + e'(H) - e'(G \cap H) \\ &= e'(G) + e'(H) - \sum_{i<j} 1(ij \in E(G)) 1(ij \in E(H)) \end{aligned}$$

e.g. number of simple subgraphs $= 2^{e'(G)}$. For $X \subseteq \{(i,j) : 1 \leq i < j \leq k\}$, let $subg(G,X)$ be the number of simple subgraphs of G that use exactly the edges X within the labelled vertices. Then

$$\#subgraphs(GH) = \sum_{X \cap Y = \emptyset} subg(G,X) subg(H,Y)$$

e.g. number of perfect matchings $pmatch(G)$ For $X \subseteq [k] = \{1, 2, \ldots, k\}$ let $pmatch(G,X)$ be the number of matchings in G that cover the unlabelled vertices and X but none of $[k] \setminus X$. Then

$$pmatch(GH) = \sum_{X \subseteq [k]} pmatch(G,X) pmatch(H, [k] \setminus X)$$

Note that it is not PSD: $k = 1$ and the submatrix induced by K_0 and K_1.

e.g. maximum clique size $\omega(G)$ consider the submatrix of $M(\omega, 0)$ induced by the cliques: it has infinite rank!

Interesting theorems about the connection matrix

multiplicative graph parameter: f is multiplicative if $f(disjointunionGandH) = f(G)f(H)$.

Theorem (6.4 in Book). If f is multiplicative, then

$$rk(f, a+b) \geq rk(f, a)rk(f, b)$$

Theorem (5.62 in Book, Lovasz and Szegedy 2012). If f is multiplicative and reflection positive defined on loop-free graphs. If $r(f, 2)$ is finite then so is $r(f, k)$ for $k \geq 2$.

Theorem 3.5 (Freedman Lovasz Welsh). If $rk(f, k)$ is finite then $f(G)$ can be computed in polynomial time for graphs with treewidth $\leq k$.

Proof. See Theorem 6.48 in the book □

Graph property, Every Property gives a parameter

Theorem (4.22 in Book). Every minor-closed graph property has finite connection rank.

Theorem (4.27 in Book) (Godlin, Kotek, Makowski 2009). Every property of graphs with no parallel edges definable by a monadic second order formula has finite connection rank.+

BREAK ...

For any weighted graph H one can define a graph parameter $f(G) = hom(G, H)$. We show it is PSD and finite rank:

recall:

$$hom(G, H) = \sum_{\psi:V(G)\to V(H)} \alpha_\psi hom_\psi(G, H)$$

where

$$\alpha_\psi = \prod_{u\in V(G)} \alpha_{\psi(u)(H)}^{\alpha_u(G)}$$

and

$$hom_\psi(G, H) = \prod_{uv\in E(G)} [\beta_{\psi(u)\psi(v)}(H)]^{\beta_{uv}(G)}$$

$$\begin{aligned} hom(G_1G_2, H) &= \sum_{\psi:V(G_1G_2)\to V(H)} \alpha_\psi hom_\psi(G, H) \\ &= \sum_{\phi:[k]\to V(H)} \sum_{\psi:V(G_1G_2)\to V(H),\psi\ \ extends\ \ \phi} \alpha_\psi hom_\psi(G, H) \end{aligned}$$

Any ψ extending ϕ can be decomposed into $\psi_1 : V(G_1) \to V(H)$ and $\psi_2 : V(G_2) \to V(H)$ such that

$$\alpha_\psi = \frac{\alpha_{\psi_1}\alpha_{\psi_2}}{\prod_{i\in[k]} \alpha_{\phi(i)}(H)}$$

hence

$$\begin{aligned} hom(G_1G_2, H) = &\sum_{\phi:[k]\to V(H)} \sum_{\psi:V(G_1G_2)\to V(H),\psi\ \ extends\ \ \phi} \alpha_\psi hom_\psi(G, H) \\ &= \sum_{\phi:[k]\to V(H)} \left(\sum_{\psi_1:V(G_1)\to V(H),\psi_1\ \ extends\ \ \phi} \frac{\alpha_{\psi_1}}{\sqrt{\alpha_\phi}} hom_{\psi_1}(G_1, H) \right) \\ &\left(\sum_{\psi_2:V(G_2)\to V(H),\psi_2\ \ extends\ \ \phi} \frac{\alpha_{\psi_2}}{\sqrt{\alpha_\phi}} hom_{\psi_2}(G_2, H) \right) \\ &= \sum_{\phi:[k]\to V(H)} g_\phi(G_1, H) g_\phi(G_2, H) \end{aligned}$$

Theorem 3.6 (Freedman Lovasz Schrijver 2004). If graph parameter f defined on loopless graphs is PSD and $rk(f, k) \leq q^k$ for all k then there exists weighted H on q vertices such that $f(G) = hom(G, H)$ for all G. (Proof is in Section 6.2.2 of the book)

Remarks:

1. Condition (b) can be replaced with another one, related to quantum graphs!
2. The number of perfect matchings satisfies (b) but not (a) so is not a homomorphism parameter.
3. The parameter "inverse of the number of simple subgraphs" is positive semidefinite but its rank grows as $2^{\binom{k}{2}}$ so is not a homomorphism parameter. However for a simple graph G, $2^{-e'(G)} = hom(G, K_1(1/2))$. (So "loopless" cannot be replaces with "simple" in theorem statement).
4. Not true if we remove "loopless": the parameter "

$$2^{-\#loop(G)}$$

" is PSD and has rank 1, but it cannot be represented as a homomorphism function. But if we consider two weights for a loop in H: one for the mappings of loops, and one for the mappings of non-loops, then the theorem remains valid.

5. extensions to directed graphs, hypergraphs, complex weights, semigroups, characterizations of $hom(F, .)$, homomorphisms to randomly weighted graphs etc. (Section 3.5 of Survey and Section 5.6 of Book)

Tightness of the bound on rank.

Two reasons it might not be tight: (1) twins: they do not change homomorphism numbers but increase $|V(H)|^k$.

(2) automorphisms: if $g_{\phi_1} = g_{\phi_2}$ then the rank goes down.

Theorem 3.8 (Lovasz'06). If H has no twins and no automorphisms then $rk(hom(.,H),k) = |V(H)|^k$ for all k.

In fact for twin-free graphs $rk(hom(.,H))$ equals the number of orbits of the automorphism group of H on ordered k-tuples of its nodes!