A Randomly Embedded Random Graph is Not a Spanner

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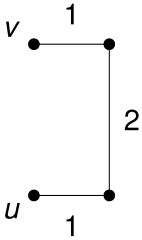
Stretch factor

The stretch factor (aka spanning ratio, dilation) of a geometric graph G is

$$\max_{u,v} \frac{d_G(u,v)}{||u - v||}$$

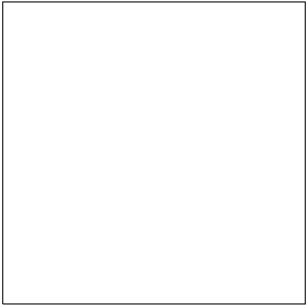
Example



stretch factor = 2

What is a randomly embedded random graph?

Let $n \in \mathbb{Z}_+$ and $p \in [0, 1]$.



$\Gamma(n, p)$

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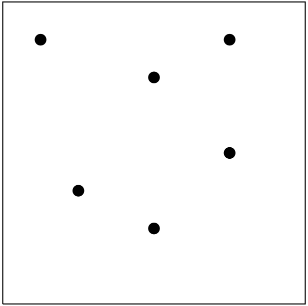

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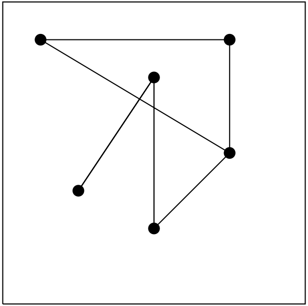

$\Gamma(n, p)$

The main result

Open Problem [O'Rourke, CCCG 2009]

When $p > \frac{\ln n}{n}$, is the stretch factor of $\Gamma(n, p)$ bounded with high probability?

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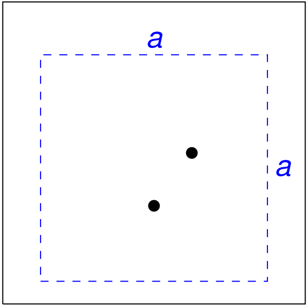
Open Problem [O'Rourke, CCCG 2009]

When $p > \frac{\ln n}{n}$, is the stretch factor of $\Gamma(n, p)$ bounded with high probability? *NO!*

Theorem

For any fixed $\epsilon, \lambda > 0$*, if* $p < 1 - \epsilon$*, then the probability that* $\Gamma(n, p)$ *has stretch factor* $< \lambda$ *approaches 0 as n grows.*

A certificate for large stretch factor



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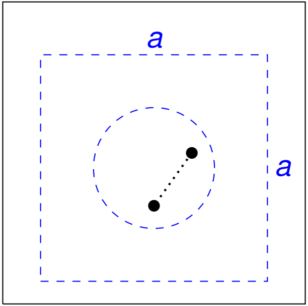

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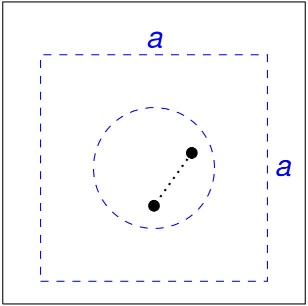

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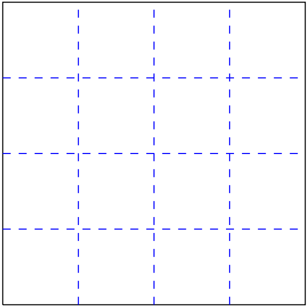
$$p_{cer} = \left(\frac{\text{circle area}}{\text{square area}}\right)^2 \times (1 - p) = \left(\frac{\pi}{4(\lambda + 1)^2}\right)^2 (1 - p) > p_0$$

A certificate for large stretch factor (cont'd)

Let A be a set of disjoint squares lying inside the unit square. Conditioned on each square in A containing exactly two vertices, the probability that $\Gamma(n, p)$ has stretch factor $< \lambda$ is at most

$$(1 - p_{cer})^{|A|}$$

Partitioning into small squares



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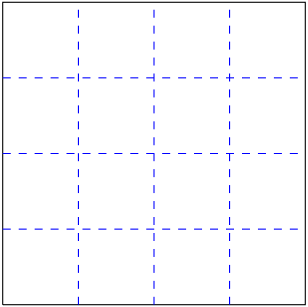

Partition into $\sim \frac{n}{2}$ smaller squares.
Let K be the number of small squares that contain exactly two vertices. Then

$$\mathbb{E}K \geq cn$$

$$\mathbf{Pr}\left[K < \frac{cn}{3}\right] \to 0$$

Wrapping up

With high probability, there are at least $cn/3$ small squares each containing exactly two vertices.

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$$(1 - p_{cer})^{cn/3} \to 0 \quad \text{as } p_{cer} > p_0$$

$\square$

Epilogue

Theorem

When $p \not\to 1$, with high probability $\Gamma(n, p)$ has unbounded stretch factor.

Epilogue

Theorem

When $p \not\to 1$*, with high probability* $\Gamma(n, p)$ *has unbounded stretch factor.*

Still true if $p \to 1$ and $n(1-p) \to \infty$.

Open Problem: What if $p \to 1$ and $n(1-p) = O(1)$?