Bidimensionality Theory: Two of the Important Results

based on a survey by Demaine and Hajiaghayi'08

Abbas Mehrabian

amehrabi@uwaterloo.ca

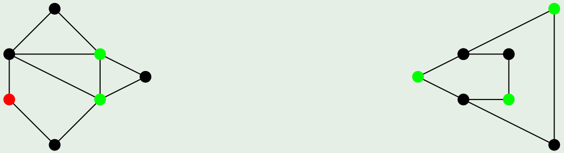
University of Waterloo

March 30th, 2011

Definition

For a graph $G = (V, E)$, a subset $S \subseteq V$ is a dominating set if every vertex not in S has a neighbour in S.

Example



$\gamma(G)$

The DominatingSet problem

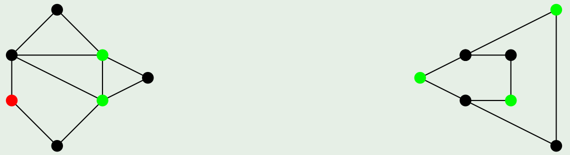
NP-hard for planar graphs

The DominatingSet Problem

Definition

For a graph $G = (V, E)$, a subset $S \subseteq V$ is a dominating set if every vertex not in S has a neighbour in S.

Example



$\gamma(G)$

The DominatingSet problem

NP-hard for planar graphs

DominatingSet in Planar Graphs

Theorem

If a tree decomposition of width w of a graph is known, then a minimum dominating set can be found in time $O(4^w n)$.

Theorem (Alber, Bodlaender, Fernau, Kloks, Niedermeier'00)

A planar graph with minimum dominating set of size k has treewidth $O(\sqrt{k})$, and a tree decomposition with treewidth $O(\sqrt{k})$ can be found in time $O(n\sqrt{k})$.

Leads to an $O(c^{\sqrt{\gamma(G)}}n)$ algorithm for the DominatingSet problem in a planar graph G.

This is called a subexponential fixed-parameter algorithm.

Bidimensionality theory is an attempt to generalize this kind of results.

DominatingSet in Planar Graphs

Theorem

If a tree decomposition of width w of a graph is known, then a minimum dominating set can be found in time $O(4^w n)$.

Theorem (Alber, Bodlaender, Fernau, Kloks, Niedermeier'00)

A planar graph with minimum dominating set of size k has treewidth $O(\sqrt{k})$, and a tree decomposition with treewidth $O(\sqrt{k})$ can be found in time $O(n\sqrt{k})$.

Leads to an $O(c^{\sqrt{\gamma(G)}}n)$ algorithm for the DominatingSet problem in a planar graph G.

This is called a subexponential fixed-parameter algorithm.

Bidimensionality theory is an attempt to generalize this kind of results.

Theorem

If a tree decomposition of width w of a graph is known, then a minimum dominating set can be found in time $O(4^w n)$.

Theorem (Alber, Bodlaender, Fernau, Kloks, Niedermeier'00)

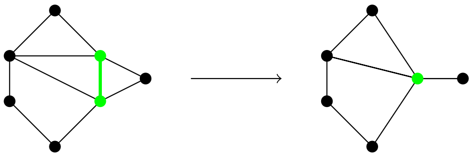
A planar graph with minimum dominating set of size k has treewidth $O(\sqrt{k})$, and a tree decomposition with treewidth $O(\sqrt{k})$ can be found in time $O(n\sqrt{k})$.

Leads to an $O(c^{\sqrt{\gamma(G)}}n)$ algorithm for the DominatingSet problem in a planar graph G.

This is called a subexponential fixed-parameter algorithm.

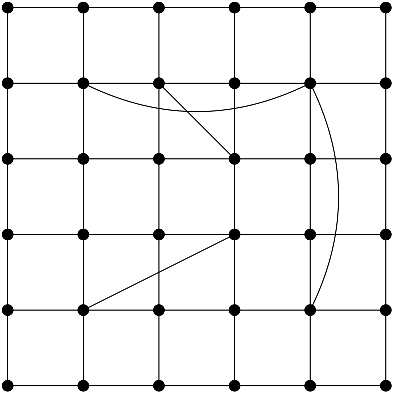
Bidimensionality theory is an attempt to generalize this kind of results.

Edge Contraction



Augmented Grid

An augmented $r \times r$ grid is an $r \times r$ grid augmented with additional edges such that each vertex is incident to C edges to nonboundary vertices of the grid.



Bidimensional Parameter

Definition

A parameter $P = P(G)$ is g(r)-bidimensional if

- $P(G)$ is at least $g(r)$ when G is an augmented $r \times r$ grid,
- $P(G)$ does not increase by contracting an edge of G.

Example

1. The number of vertices of G is r^2-bidimensional.
2. The minimum size of a dominating set of G is $\Omega(r^2)$-bidimensional.
3. The treewidth of G is r-bidimensional.

Bidimensional Parameter

Definition

A parameter $P = P(G)$ is g(r)-bidimensional if

- $P(G)$ is at least $g(r)$ when G is an augmented $r \times r$ grid,
- $P(G)$ does not increase by contracting an edge of G.

Example

1. The number of vertices of G is r^2-bidimensional.
2. The minimum size of a dominating set of G is $\Omega(r^2)$-bidimensional.
3. The treewidth of G is r-bidimensional.

Bidimensional Parameter

Definition

A parameter $P = P(G)$ is g(r)-bidimensional if

- $P(G)$ is at least $g(r)$ when G is an augmented $r \times r$ grid,
- $P(G)$ does not increase by contracting an edge of G.

Example

1. The number of vertices of G is r^2-bidimensional.
2. The minimum size of a dominating set of G is $\Omega(r^2)$-bidimensional.
3. The treewidth of G is r-bidimensional.

Bidimensional Parameter

Definition

A parameter $P = P(G)$ is g(r)-bidimensional if

- $P(G)$ is at least $g(r)$ when G is an augmented $r \times r$ grid,
- $P(G)$ does not increase by contracting an edge of G.

Example

1. The number of vertices of G is r^2-bidimensional.
2. The minimum size of a dominating set of G is $\Omega(r^2)$-bidimensional.
3. The treewidth of G is r-bidimensional.

A General Parameter-Treewidth Bound

Theorem (The General Parameter-Treewidth Bound)

If P is a $g(r)$-bidimensional parameter, then for any planar G,

$$treewidth(G) = tw(G) = O(g^{-1}(P(G)))$$

In particular, if $g(r) = \Omega(r^2)$, then $tw(G) = O(\sqrt{P(G)})$.

Proved by: Demaine and Hajiaghayi'05 based on the work of Demaine, Fomin, Hajiaghayi, and Thilikos'04

Example

1. If $P(G) = |V(G)|$, which is r^2-bidimensional, then any planar G has $tw(G) = O(\sqrt{|V(G)|})$.
2. If $P(G) = \gamma(G)$, which is $\Omega(r^2)$-bidimensional, then any planar G has $tw(G) = O(\sqrt{|\gamma(G)|})$.

A General Parameter-Treewidth Bound

Theorem (The General Parameter-Treewidth Bound)

If P is a $g(r)$-bidimensional parameter, then for any planar G,

$$treewidth(G) = tw(G) = O(g^{-1}(P(G)))$$

In particular, if $g(r) = \Omega(r^2)$, then $tw(G) = O(\sqrt{P(G)})$.

Proved by: Demaine and Hajiaghayi'05 based on the work of Demaine, Fomin, Hajiaghayi, and Thilikos'04

Example

1. If $P(G) = |V(G)|$, which is r^2-bidimensional, then any planar G has $tw(G) = O(\sqrt{|V(G)|})$.
2. If $P(G) = \gamma(G)$, which is $\Omega(r^2)$-bidimensional, then any planar G has $tw(G) = O(\sqrt{|\gamma(G)|})$.

A General Parameter-Treewidth Bound

Theorem (The General Parameter-Treewidth Bound)

If P is a $g(r)$-bidimensional parameter, then for any planar G,

$$treewidth(G) = tw(G) = O(g^{-1}(P(G)))$$

In particular, if $g(r) = \Omega(r^2)$, then $tw(G) = O(\sqrt{P(G)})$.

Proved by: Demaine and Hajiaghayi'05 based on the work of Demaine, Fomin, Hajiaghayi, and Thilikos'04

Example

1. If $P(G) = |V(G)|$, which is r^2-bidimensional, then any planar G has $tw(G) = O(\sqrt{|V(G)|})$.
2. If $P(G) = \gamma(G)$, which is $\Omega(r^2)$-bidimensional, then any planar G has $tw(G) = O(\sqrt{|\gamma(G)|})$.

The general parameter-treewidth bound + algorithms for computing approximately optimal tree decompositions, give

Theorem

If P is an $\Omega(r^2)$-bidimensional parameter, which can be computed in a planar graph G in time $O\left(d^w n^{O(1)}\right)$ given a tree decomposition of G of width w, then there is an algorithm for computing $P(G)$ on a planar graph G with running time $O\left(c^{\sqrt{P(G)}} n^{O(1)}\right)$.

A fixed parameter algorithm, and subexponential in P.

Example 1

There is an algorithm for the DominatingSet problem in planar graphs G with running time $O\left(c^{\sqrt{\gamma(G)}} n^{O(1)}\right)$.

The general parameter-treewidth bound + algorithms for computing approximately optimal tree decompositions, give

Theorem

If P is an $\Omega(r^2)$-bidimensional parameter, which can be computed in a planar graph G in time $O\left(d^w n^{O(1)}\right)$ given a tree decomposition of G of width w, then there is an algorithm for computing $P(G)$ on a planar graph G with running time $O\left(c^{\sqrt{P(G)}} n^{O(1)}\right)$.

A fixed parameter algorithm, and subexponential in P.

Example 1

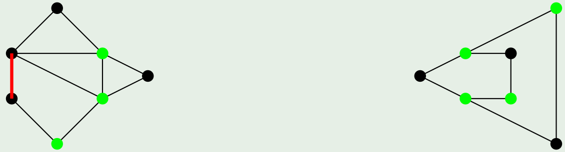
There is an algorithm for the DominatingSet problem in planar graphs G with running time $O\left(c^{\sqrt{\gamma(G)}} n^{O(1)}\right)$.

Example 2: The VertexCover Problem

Definition

A vertex cover in graph $G = (V, E)$ is a subset $S \subseteq V$ such that for every edge e, at least one endpoint of e is in S. Let $vc(G)$ be the size of a minimum vertex cover in G.

Example



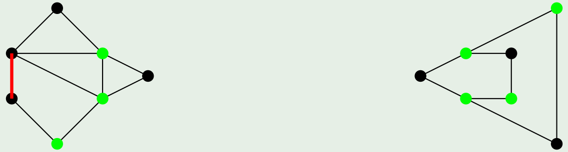
$vc(G)$ is $\Omega(r^2)$-bidimensional, so we find an $O\left(c^{\sqrt{vc(G)}} n^{O(1)}\right)$ algorithm for planar graphs.

Example 2: The VertexCover Problem

Definition

A vertex cover in graph $G = (V, E)$ is a subset $S \subseteq V$ such that for every edge e, at least one endpoint of e is in S. Let $vc(G)$ be the size of a minimum vertex cover in G.

Example



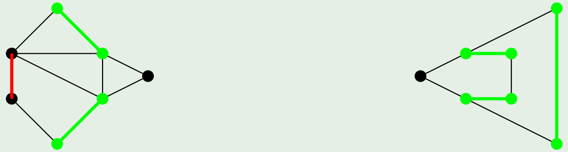
$vc(G)$ is $\Omega(r^2)$-bidimensional, so we find an $O\left(c^{\sqrt{vc(G)}} n^{O(1)}\right)$ algorithm for planar graphs.

Example 3: The MaximalMatching Problem

Definition

A matching M is a maximal matching if it cannot be extended by adding a new edge. Let $mm(G)$ be the size of a maximal matching of G having minimum size.

Example



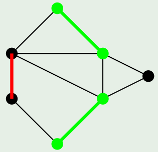
$mm(G)$ is $\Omega(r^2)$-bidimensional, so we get an $O\left(c^{\sqrt{mm(G)}} n^{O(1)}\right)$ algorithm for finding a minimum maximal matching in planar graphs.

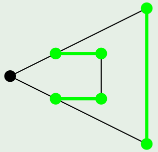
Example 3: The MaximalMatching Problem

Definition

A matching M is a maximal matching if it cannot be extended by adding a new edge. Let $mm(G)$ be the size of a maximal matching of G having minimum size.

Example





$mm(G)$ is $\Omega(r^2)$-bidimensional, so we get an $O\left(c^{\sqrt{mm(G)}} n^{O(1)}\right)$ algorithm for finding a minimum maximal matching in planar graphs.

DominatingSet in Planar Graphs

Theorem

If a tree decomposition of width w of a graph is known, then a minimum dominating set can be found in time $O(4^w n)$.

Theorem (Alber, Bodlaender, Fernau, Kloks, Niedermeier'00)

A planar graph with minimum dominating set of size k has treewidth $O(\sqrt{k})$, and a tree decomposition with treewidth $O(\sqrt{k})$ can be found in time $O(n\sqrt{k})$.

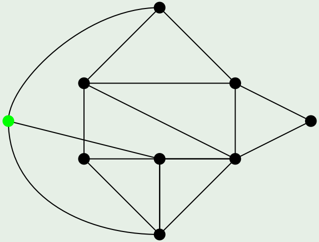
Leads to an $O(c^{\sqrt{\gamma(G)}}n)$ algorithm for the DominatingSet problem in a planar graph G.

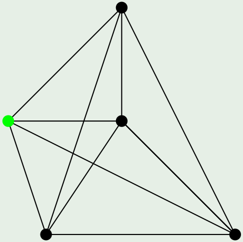
Apex Graphs

Definition

Graph G is an apex graph if it has a vertex whose removal results in a planar graph.

Example





Apex-Minor-Free Graphs

Definition

Let H be a fixed graph. Graph G is H-minor-free if one cannot obtain H from G by a sequence of vertex and edge deletions, and edge contractions.

E.g. every planar graph is K_5-minor-free.

Theorem

The parameter-treewidth bound and resulting subexponential fixed-parameter algorithms indeed hold for all H-minor-free graphs, where H is a fixed apex graph.

Apex-Minor-Free Graphs

Definition

Let H be a fixed graph. Graph G is H-minor-free if one cannot obtain H from G by a sequence of vertex and edge deletions, and edge contractions.

E.g. every planar graph is K_5-minor-free.

Theorem

The parameter-treewidth bound and resulting subexponential fixed-parameter algorithms indeed hold for all H-minor-free graphs, where H is a fixed apex graph.

Eppstein's Diameter-Treewidth Bound

Theorem (Eppstein'00)

Let H be an apex graph. Every H-minor-free graph of diameter D has treewidth

$$tw(G) = 2^{2^{O(D)}}$$

Leads to $(1+\epsilon)$-approximation algorithms with running time $2^{2^{2^{O(1/\epsilon)}}} n$ for finding minimum independent set in apex-minor-free graphs. (Similar to Baker's proof)

The diameter is $\Omega(\log r)$-bidimensional, so the general parameter-treewidth bound gives

$$tw(G) = 2^{O(D)}$$

Theorem (Eppstein'00)

Let H be an apex graph. Every H-minor-free graph of diameter D has treewidth

$$tw(G) = 2^{2^{O(D)}}$$

Leads to $(1+\epsilon)$-approximation algorithms with running time $2^{2^{2^{O(1/\epsilon)}}} n$ for finding minimum independent set in apex-minor-free graphs. (Similar to Baker's proof)

The diameter is $\Omega(\log r)$-bidimensional, so the general parameter-treewidth bound gives

$$tw(G) = 2^{O(D)}$$

Eppstein's Diameter-Treewidth Bound

Theorem (Eppstein'00)

Let H be an apex graph. Every H-minor-free graph of diameter D has treewidth

$$tw(G) = 2^{2^{O(D)}}$$

Leads to $(1+\epsilon)$-approximation algorithms with running time $2^{2^{2^{O(1/\epsilon)}}} n$ for finding minimum independent set in apex-minor-free graphs. (Similar to Baker's proof)
The diameter is $\Omega(\log r)$-bidimensional, so the general parameter-treewidth bound gives

$$tw(G) = 2^{O(D)}$$

A Better Diameter-Treewidth Bound

Theorem (The Optimal Diameter-Treewidth Bound)

Let H be an apex graph. Every H-minor-free graph of diameter D has treewidth $O(D)$.

Proved by: Demaine and Hajiaghayi'04

Leads to $(1+\epsilon)$-approximation algorithms with running time

$$O(c^{\frac{1}{\epsilon}} n)$$

for finding maximum independent set, minimum dominating set, and minimum vertex cover in apex-minor-free graphs.

A Better Diameter-Treewidth Bound

Theorem (The Optimal Diameter-Treewidth Bound)

Let H be an apex graph. Every H-minor-free graph of diameter D has treewidth $O(D)$.

Proved by: Demaine and Hajiaghayi'04

Leads to $(1+\epsilon)$-approximation algorithms with running time

$$O(c^{\frac{1}{\epsilon}} n)$$

for finding maximum independent set, minimum dominating set, and minimum vertex cover in apex-minor-free graphs.

Minor-Bidimensional Parameters (Definition)

Definition (g(r)-bidimensional)

A parameter $P = P(G)$ is g(r)-bidimensional if

- $P(G)$ is at least $g(r)$ when G is an augmented $r \times r$ grid,
- $P(G)$ does not increase by contracting an edge of G.

Definition (g(r)-minor-bidimensional)

A parameter $P = P(G)$ is g(r)-minor-bidimensional if

- $P(G)$ is at least $g(r)$ when G is an $r \times r$ grid,
- $P(G)$ does not increase by contracting an edge, or deleting an edge or a vertex of G.

Minor-Bidimensional Parameters (Examples)

Definition (g(r)-minor-bidimensional)

A parameter $P = P(G)$ is g(r)-minor-bidimensional if

- $P(G)$ is at least $g(r)$ when G is an $r \times r$ grid,
- $P(G)$ does not increase by contracting an edge, or deleting an edge or a vertex of G.

Example

1. The minimum dominating set and diameter are NOT minor-bidimensional (delete an edge).
2. The minimum vertex cover and minimum maximal matching are $\Omega(r^2)$-minor-bidimensional.

Instead of restricting to apex-minor-free graphs, the discussed fixed-parameter subexponential algorithms exist for H-minor-free graphs, where H is any fixed graph.

Thank You!

Any Questions?