Introduction

These are the notes of the talk I gave on May 10th 2011 in University of Waterloo. The aim is to describe the main contributions and ideas of the paper

> Computing the Independence Number of Intersection Graphs, by Jacob Fox and Janos Pach (SODA'11).

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Define: String graphs, k-intersecting family of curves n vertices, m edges, and x crossings

We will give an n^{ϵ}-approximation algorithm and subexponential exact algorithms for independent set of k-intersecting string graphs.

Based on several previous papers of the same authors (see [2, 3]).

Separators

Theorem 1 (Theorems 1.1 and 2.5 in [3])**.** *A string graph has an* $O(m^{3/4}\sqrt{\log m})$*-separator and an* $O(\Delta\sqrt{m}\log m)$*-separator.*

Conjecture 1.2 in [3]
A string graph has an $O(\sqrt{m})$-separator.

Lemma 2 (Lemma 2.2 in [1], Theorem 3 in [2])**.** *Intersection graph of a collection of curves with a total of* x *intersections has a separator of size* $O(\sqrt{x})$*, which can be found efficiently.*

Proof. Build a weighted planar graph P on the set of intersection points (so $O(x)$ vertices), with $w(v) = 1/d(\gamma_1) + 1/d(\gamma_2)$ if v is on γ_1 and γ_2. Use Lipton-Tarjan's separator theorem to find an $O(\sqrt{x})$-separator S, separating $V(P)$ into P_1 and P_2. Let V_i be curves with all points lying in P_i, and V_0 be curves with at least one point in S. Then V_0 is an $O(\sqrt{x})$-separator for G. □

Subexponential Algorithms

Theorem 3 (Theorem 1.3 in [1]). *Fix k. Given a k-intersecting string graph G, we can compute a maximum independent set of G in time* $\exp(n^{2/3} polylog(n))$.

Proof. Let $g(n) = \exp(\Theta(n^{2/3}))$ and $d = n^{1/3}$.

Case 1: $\Delta \geq d$. Let v be a vertex with degree $\geq d$. Compute max ind sets of $G - v$ and $G - N(v)$. Running time is at most

$$g(n-1) + g(n-d) + O(n^2) \leq g(n)$$

Case 2: $\Delta < d$. Then $m \leq nd/2$ so $x \leq ndk/2$. By Lemma 2 there is a separator S of size $s = O(\sqrt{x}) = O(\sqrt{n}d) = O(n^{2/3})$. Find it in time

$$n^s = n^{n^{2/3}} = exp(n^{2/3} \log n) = exp(\Theta(n^{2/3})) < g(n)/2$$

Try all subsets of S, together with max ind sets of both sides. The running time is

$$O\left(2^s n^2 g(2n/3)\right) < g(n)/2$$

□

Remark. The polylog in the exponent can be removed using a different idea (see Theorem 1.3 in [1]).

Theorem 4 (Theorem 1.2 in [1]). *A maximum independent set of a string graph can be computed in time* $\exp(n^{4/5} polylog(n))$.

Approximation Algorithm

Theorem 5 (Lemma 2.1 in [1], Theorem 1.5 in [3])**.** *For any k, there is a constant c_k such that: Every k-intersecting graph contains $K_{t,t}$ as a subgraph, with $t = c_k m/n$. Such a $K_{t,t}$ can be detected quickly.*

Proof. Later. □

Theorem 6 (Theorem 1.1 in [1])**.** *Fix k and ϵ. We can compute an n^ϵ-approximation of the max independent set of a k-intersecting string graph G in time $n^{O\left((4/\epsilon)^{2/\epsilon}\right)}$*

Proof. See Page 3 of [1]. □

Proof of Theorem 5

We will prove a weaker statement.

Theorem 7. *For any k and t, there is a constant $c_{k,t}$ such that: Every $K_{t,t}$-free k-intersecting string graph has $m \leq c_{k,t}n$.*

Remark. If we had $c_{k,t} = t/c_k$ then we would get Theorem 5 but the actual dependence of $c_{k,t}$ on t is much worse than linear.

Proof. The proof needs the following two results.

Kovari-Sos-Turan If $n \geq 2t$, then any $K_{t,t}$-free graph has $m \leq n^{2-1/t}$.

Lipton-Rose-Tarjan Let S be a family of graphs, closed under taking subgraphs, such that any $G \in S$ has an $O(n/(\log n)^2)$-separator. Then any $G \in S$ has $m = O(n)$.

The number of intersection points is $O(km)$, so by Lemma 2 G has a $O(\sqrt{km})$-separator. Since G is $K_{t,t}$-free, by KST this is an $O(\sqrt{k}n^{1-1/2t})$-separator. So, by LRT, $m = O(n)$. □

Theorem 8 (Theorem 1.3 in [3])**.** *Every $K_{t,t}$-free string graph has $m \leq t^{O(\log \log t)}n$, and the conjecture is that the correct bound is $m \leq O(nt \log t)$.*

References

[1] J. Fox and J. Pach, Computing the Independence Number of Intersection Graphs, SODA 2011.

[2] J. Fox and J. Pach, Separator Theorems and Turan-type Results for Planar Intersection Graphs, Advances in Mathematics 219 (2008), 1070–1080.

[3] J. Fox and J. Pach, A Separator Theorem for String Graphs and Its Applications, Combin. Probab. Comput. 19 (2010), 371–390.