The Setup

$V = \{1, 2, \ldots, n\}$ is a set of *elements*, also called *points*.

$\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ is a family of m subsets of V.

$v_1, \ldots, v_m \in \mathbb{R}^n$ are the indicator vectors for the sets.

A *complete colouring* is a function $\chi : V \to \{-1, +1\}$.

A *partial colouring* is a function $\chi : V \to [-1, +1]$.

The *discrepancy* of a S_j is $\mathrm{disc}(S_j) = |\sum_{i \in S_j} \chi(i)| = |\langle \chi, v_j \rangle|$.

The *discrepancy* of a colouring χ is

$$\chi(S) = \max\{\mathrm{disc}(S) : S \in \mathcal{S}\} = \max\{|\langle \chi, v_j \rangle| : j = 1, \ldots, m\}.$$

The *discrepancy* of a set system $(V, \mathcal{S})$ is the minimum discrepancy of a complete colouring of it.

Problem. Find upper bounds for discrepancy.

Results:

	existential	algorithmic	algorithmic
random colouring	$O\left(\sqrt{n\log m}\right)$	$O\left(\sqrt{n\log m}\right)$	
General Setting	$O\left(\sqrt{n\log\frac{2m}{n}}\right)$ Spencer'85	$O\left(\sqrt{n}\log\frac{2m}{n}\right)$ Bansal'10	$O\left(\sqrt{n\log\frac{2m}{n}}\right)$ this paper
Maximum Degree d	$O\left(\sqrt{d\log n}\right)$ Banaszczyk'98	$O\left(\sqrt{d}\log n\right)$ Bansal'10	$O\left(\sqrt{d}\log n\right)$ this paper

Inapproximability of Discrepancy: Charikar, Newman and Nikolov proved that there are systems with $m = O(n)$ sets, such that no polynomial algorithm can distinguish whether the discrepancy is 0 or $\Omega(\sqrt{n})$, unless $P = NP$.

Theorem 1. Let $m \geq n$. There is a randomized algorithm with running time $O((m+n)^3 \log^5(mn))$ that with probability at least $1/2$ constructs a (complete) colouring with discrepancy $13\sqrt{n \log_2(2m/n)}$.

The Partial Colouring Lemma. Let $m \geq n$, $v_1, \ldots, v_m \in \mathbb{R}^n$, and $x_0 \in [-1,1]^n$. Let $c_1, \ldots, c_m \geq 0$ be such that

$$\sum_{j=1}^{m} \exp(-c_j^2/16) \leq n/16$$

Let $\delta \in (0, 0.1)$ be some approximation parameter. There exists a randomized algorithm with running time $poly(m, n, 1/\delta)$ that with probability ≥ 0.1 finds a point $x \in [-1,1]^n$ such that

1. $|\langle x - x_0, v_j \rangle| \leq c_j \|v_j\|_2$
2. $|x_i| \geq 1 - \delta$ for at least $n/2$ indices $i \in [n]$.

Proof of Theorem 1. Let $\delta = 1/n$ and $\alpha(m,n) = 8\sqrt{\log(2m/n)}$, and $x_0 = 0^n$. Note $m \exp\left(-\alpha(m,n)^2/16\right) \leq n/16$, so with probability at least 0.1 we find $x_1 \in [-1,1]^n$ with

$$\forall j \ |\langle v_j, x_1 \rangle| \leq \alpha(m,n)\sqrt{n}, \qquad |\{i : |(x_1)_i| \geq 1 - \delta\}| \geq n/2$$

Repeat $\log n$ times to get sufficiently large success probability.

Assume that n_1 points have not been coloured, let $x_1 \in \mathbb{R}^{n_1}$ be the vector of uncoloured points. Apply theorem again for the new vectors $v'_1, \ldots, v'_m \in \mathbb{R}^{n_1}$ to get another vector $x_2 \in [-1,1]^{n_1}$ such that

$$\forall j \ |\langle v'_j, x_2 \rangle| \leq \alpha(m,n_1)\sqrt{n_1}, \qquad |\{i : |(x_2)_i| \geq 1 - \delta\}| \geq n_1/2$$

Repeat this $2\log n$ times until all points have been coloured and you get a vector $x \in \mathbb{R}^n$. Then $|x_i| \geq 1 - \delta$ for all $i \in [n]$, and for each $j \in [m]$,

$$|\langle v_j, x \rangle| \leq \sqrt{n}\alpha(m,n) + \sqrt{n_1}\alpha(m,n_1) + \cdots = O\left(\sqrt{n \log(2m/n)}\right)$$

□

An outline of the algorithm ...
BREAK !!

(I) Let $N(\mu, \sigma^2)$ denote Gaussian distribution with mean μ and variance σ^2.

(II) If $G_1 \sim N(\mu_1, \sigma_1^2)$ and $G_2 \sim N(\mu_2, \sigma_2^2)$ then

$$t_1 G_1 + t_2 G_2 \sim N(t_1\mu_1 + t_2\mu_2, t_1^2\sigma_1^2 + t_2^2\sigma_2^2)$$

(III) Let $V \subseteq \mathbb{R}^n$ be a linear subspace, and $\{v_1, \ldots, v_d\}$ be an orthonormal basis for it, and $G_1, \ldots, G_d \sim N(0, 1)$ be independent. Then $G = G_1 v_1 + \cdots + G_d v_d$ is distributed as $N(V)$.

(IV) If $G \sim N(V)$, then for any $u \in \mathbb{R}^n$ we have $\langle G, u \rangle \sim N(0, \sigma^2)$, with $\sigma \leq \|u\|$.

(V) If $G \sim N(V)$, then $E[\|G\|^2] = dim(V)$.

(VI) Let $H \sim N(0, 1)$. For all $\lambda > 0$, $Pr[|H| \geq \lambda] \leq 2\exp(-\lambda^2/2)$.

(VII) Let $X_1, \ldots, X_T$ be random variables, and $Y_1, \ldots, Y_T$ be random variables such that Y_i is a function of X_i. Suppose that for all $x_1, \ldots, x_{i-1}$, $Y_i|(X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$ is Gaussian with mean zero and variance at most one (possible depending on $x_1, \ldots, x_{i-1}$). Then for any $\lambda > 0$,

$$Pr[|Y_1 + \cdots + Y_T| \geq \lambda] \leq 2\exp(\lambda^2/2T^2)$$

Write down the algorithm from the article.

Observations.

1. Polynomial running time.
2. C^t_{var} and C^t_{disc} are increasing, in particular, $dim(V_T) = \min\{dim(V_1), \ldots, dim(V_T)\}$.

Claim 1. With high probability, $X^0, \ldots, X^T \in P$.

Proof. Let E^t be the event that at time t we go out of P for the first time. If E^t happens then we must have $|\langle X^t - X^{t-1}, w\rangle| \geq \delta$ for some $w \in \{e_1, \ldots, e_n, v_1, \ldots, v_m\}$. But $\langle U^t, w\rangle \sim N(0, \sigma^2)$ for some $\sigma \leq \|w\| = 1$ by (IV). So by (VI),

$$Pr[E_t] \leq 2\exp(-(\delta/\gamma)^2/2) = 2\,(\gamma/mn)^{C/2}$$

By union bound,

$$Pr[\exists i : X^i \notin P] \leq T \times (2m) \times 2\,(\gamma/mn)^{C/2} = o(1)$$

□

Proof Intuition : We have

$$C^t_{disc} + C^t_{var} + dim(V^t) \geq n$$

Claim 2. Claim 14 from the article.
Claim 3. $E[\|X^T\|^2] \leq n$.

Proof. Fix some $i \in [n]$. If $i \notin C^T_{var}$ then $E[\|X^T\|^2] \leq 1$. Let t be the the first time variable $|X^t_i| > 1$, condition on X^{t-1}.

$$E[(X^t_i)^2] = E[(X^{t-1}_i + \gamma U^t_i)^2] = (X^{t-1}_i)^2 + \gamma^2 E[(U^t_i)^2] \leq 1 - \delta + \gamma^2 \leq 1$$

by (IV). □

Claim 4. Claim 16 from the article.

Proof of The Partial Colouring Lemma. By Claim 4 and since $C_{var} \leq n$, $P[|C^T_{var}| \geq n/2] \geq 0.12$. The probability that we go out of P is < 0.1 by Claim 1. Hence with probability 0.1 we colour at least half of the points and do not go out of P. □