This is a summary of the talk I gave on September 25th of 2013 in University of Waterloo. The aim is to explain the paper: BASIC NETWORK CREATION GAMES, by Alon, Demaine, Hajiaghayi and Leighton, SPAA 2010, SiDMa 2013. Pictures below are taken from that paper.

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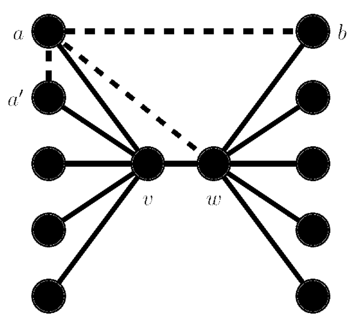
1. Network creation games
2. Nash equilibria
3. (*) A simple undirected graph is a sum (max) equilibrium graph if for every edge xy and every non-edge xz, deleting xy and adding xz does not decrease the total sum (maximum) of distances from x to all other vertices.
4. Main ideas in this paper: remove the "creation cost", considers swap equilibria, so best response not NP-hard
5. (*) Parameter of study: Largest diameter of an equilibrium graph ... related to price of anarchy
6. (*) *cost of vertex u*: sum of distances from u to other vertices
7. n is the number of vertices, we consider connected graphs and all logarithms are in base 2

KEEP ON BOARD(*)

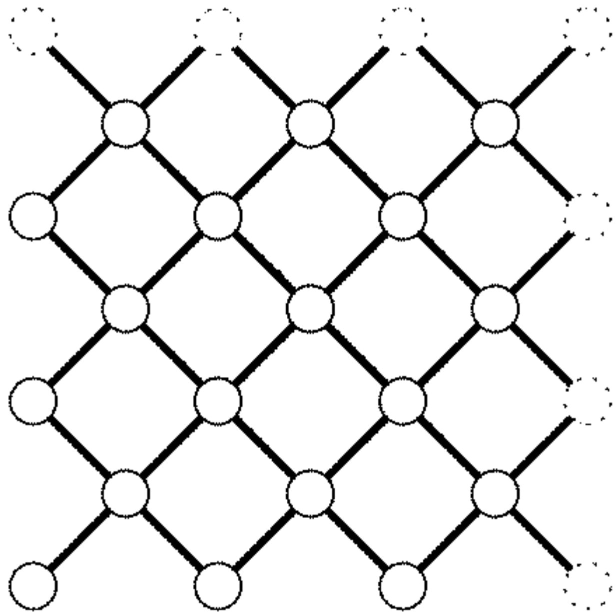
	SUM	MAX
trees	2	3
general	$3 \leq \ldots\ldots \leq 2^{O(\sqrt{\log n})}$	$\Omega(\sqrt{n})$

Theorem 1. *If a sum eq. graph is a tree, then the diameter < 3.*

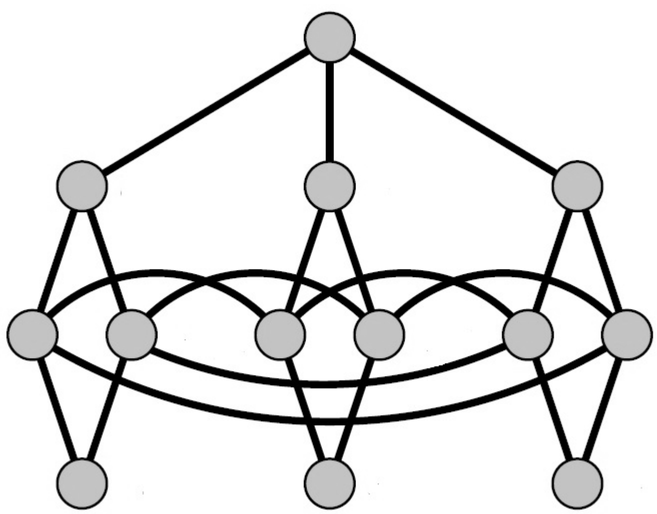
Proof. Consider a path of length 3, expressway, swapping, etc. □



A tree in max eq. with diameter 3. Three possible edges to add. Only aw decreases the maximum distance of one of its endpoints, but it actually does not!



A $\Theta(\sqrt{n})$-diameter max equilibrium graph



A diameter-3 sum equilibrium graph

Lemma 1. *Let $G = (V, E)$ be a graph, $u, v, x \in V$ with $uv \notin E$. Assume that adding edge uv to G decreases cost of u by s, where $s > n\,\mathrm{d}(x, u)$. Then adding edge xv to G decreases cost of x by at least $s - n\,\mathrm{d}(x, u)$.*

Proof. For vertices a, b, let $\mathrm{improve}_a(b)$ be the amount a gets closer to b by adding av. Let $\mathrm{d}^{new}(a, b)$ be the distance between a and b in $G \cup av$. Let W be the set of vertices w with $\mathrm{improve}_u(w) > 0$. For all $w \in W$ we have

$$\begin{aligned}\mathrm{improve}_x(w) = \mathrm{d}(x, w) - \mathrm{d}^{new}(x, w) \geq &(\mathrm{d}(u, w) - \mathrm{d}(u, x)) - (1 + \mathrm{d}(v, w))\\ =&\,\mathrm{d}(u, w) - (1 + \mathrm{d}(v, w)) - \mathrm{d}(u, x)\\ =&\,\mathrm{improve}_u(w) - \mathrm{d}(u, x).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{w\in V} \mathrm{improve}_x(w) \geq \sum_{w\in W} \mathrm{improve}_x(w) \geq &\sum_{w\in W} [\mathrm{improve}_u(w) - \mathrm{d}(u, x)]\\ =&\,s - |W|\,\mathrm{d}(u, x) \geq s - n\,\mathrm{d}(u, x),\end{aligned}$$

and the proof is complete. □

Theorem 2. *All connected sum equilibrium graphs have diameter $2^{O(\sqrt{\log n})}$.*

Let $G = (V, E)$ be a sum equilibrium graph with diameter $> 2 \log n + 2$.
Claim 1. For each $u \in V$ there exists edge xy with $\mathrm{d}(u, x) \leq 1 + \log n$ whose deletion increases cost of x by $\leq 2n(1 + \log n)$.

Proof of Claim 1. Consider BFS tree rooted at u from level 0 (just u) to level $1 + \log n$.

If there is a cross-edge xy, done.

WMA There is no cross-edge.
T_v := subtree rooted at v.
ground := level $1 + \log n$
grounded vertex : a vertex with a descendent at ground
$gd(v)$:= distance between v and ground.

If u is not grounded the diameter is $\leq 2 \log n + 2$, done.

WMA u is grounded. We show this case is impossible!
Claim 1.1. If $v \neq u$ is grounded then $|T_v| \geq 2^{gd(v)}$.

Proof of Claim 1.1. By induction on $gd(v)$. Easy for $gd(v) = 0$. Assume v has > 1 grounded children a and b. Then

$$|T_v| \geq 1 + |T_a| + |T_b| \geq 1 + 2 \times 2^{gd(a)} = 1 + 2^{gd(v)}$$

WMA v has 1 grounded child a so v is a cut vertex.
p := parent of v. If p swaps edge pv with pa, the change in its cost is:

$$-|T_a| + |T_v| - |T_a| \geq 0$$

so

$$|T_v| \geq 2|T_a| \geq 2 \times 2^{gd(a)} = 2^{gd(v)}$$

□

Let v be a grounded child of u. Then by Claim 1.1, $|T_u| > |T_v| \geq 2^{\log n}$, case impossible! □

Claim 2. Let $uv \notin E$. Joining u and v decreases cost of u by $\leq 6n \log n$.

Proof. Suppose uv is a counterexample. Use Claim 1 to find xy. If x swaps xy with xv, its cost increases by $\leq 2n(1 + \log n) \leq 3n \log n$ (Claim 1); since $\mathrm{d}(x, u) \leq 1 + \log n$, its cost decreases by $\geq 6n \log n - n(1 + \log n) \geq 4n \log n$ (Lemma 1). □

$B(u, k) :=$ set of vertices with distance at most k from u,
$f(k) := \min_{u \in V} |B(u, k)|$.
Claim 3.

$$f(4k) \geq \min \left\{ \frac{n}{2}\,,\, \frac{kf(k)}{24 \log n} \right\}.$$

Proof. Let $u \in V$ and WMA $B(u, 4k) < n/2$. Then $B(u, 3k) < n/2$.
$S :=$ a maximal set of vertices at distance $3k$ from u such that the distance between any pair of vertices in S is $\geq 2k + 1$.
For every $v \notin B(u, 3k)$, distance of v from some $s \in S$ is $\leq d(u, v) - k$.
So there exists $s_0 \in S$ and $A \subseteq V \setminus B(u, 3k)$ so that $|A| \geq n/2|S|$ and $\forall a \in A$, $\mathrm{d}(a, s_0) \leq d(a, u) - k$.
Joining u and s_0 decreases cost of u by $\geq (k - 1)n/2|S| \geq kn/4|S|$.
This is $\leq 6n \log n$ by Claim 2, so $|S| \geq k/24 \log n$.
Balls $\{B(s, k) : s \in S\}$ are disjoint and lie in $B(u, 4k)$, hence $|B(u, 4k)| \geq kf(k)/24 \log n$. □

Proof of Theorem 2.

$$f(2^{\sqrt{\log n}}) \geq 2^{\sqrt{\log n}}$$

because G is connected.
$0 \leq \sigma :=$ smallest integer for which $f(2^{\sqrt{\log n}} 4^\sigma) \geq n/2$. By Claim 3, for every $1 \leq i < \sigma$,

$$\frac{f(2^{\sqrt{\log n}} 4^i)}{f(2^{\sqrt{\log n}} 4^{i-1})} \geq \frac{2^{\sqrt{\log n}} 4^{i-1}}{24 \log n} = 2^{\Omega(\sqrt{\log n})}.$$

So, for all $1 \leq i < \sigma$,

$$f(2^{\sqrt{\log n}} 4^i) = 2^{\Omega(i\sqrt{\log n})}.$$

But, since

$$f(2^{\sqrt{\log n}} 4^{\sigma-1}) \leq n/2 = 2^{\log n - 1},$$

we have $\sigma = O(\sqrt{\log n})$. Since $f(2^{\sqrt{\log n}} 4^\sigma) \geq n/2$. for any two vertices u and v, the distance between u and v is at most

$$1 + 2 \times 2^{\sqrt{\log n}} 4^\sigma = 2^{O(\sqrt{\log n})},$$

□

The authors conjecture all sum equilibrium graphs have polylogarithmic diameter.

A graph is ϵ-distance-uniform if there exists r such that, for every vertex v, the number of vertices at distance exactly r from v is $\geq (1-\epsilon)n$.

Theorem 3. *Let G be a sum equilibrium graph with n vertices and diameter $g > 2\log n$. Then there exists an $\epsilon > 0$ and an ϵ-distance-uniform graph G' with n vertices and diameter $\Theta(\epsilon g / \log^2 n)$.*

Conjecture. Distance-uniform-graphs have logarithmic diameter.